Biometrika (2021), **103**, 1, *pp.* 1–17
Printed in Great Britain Advance Access publication on 31 July 2018

Multiplicative Structural Nested Mean Model for Zero-Inflated Outcomes

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SUMMARY

Zero-inflated nonnegative outcomes are common in many applications. In this work, motivated by freemium mobile game data, we propose a class of multiplicative structural nested mean models for zero-inflated nonnegative outcomes, which flexibly describes the joint effect of a sequence of treatments in the presence of time-varying confounders. The proposed estimator solves a doubly robust estimating equation, where the nuisance functions, namely, the propensity score and conditional outcome means given confounders, are estimated parametrically or nonparametrically. To improve the accuracy, we leverage the characteristic of zero-inflated outcomes by estimating the conditional means in two parts, that is, separately modeling the probability of having positive outcomes given confounders and the mean outcome conditional on its being positive and confounders. We show that the proposed estimator is consistent and asymptotically normal as either the sample size or the follow-up time goes to infinity. Moreover, the typical sandwich formula can be used to estimate the variance of treatment effect estimators consistently, without accounting for the variation due to estimating nuisance functions. Simulation studies and an application to a freemium mobile game dataset are presented to demonstrate the empirical performance of the proposed method and support our theoretical findings.

Some key words: Bidirectional Asymptotics; Timewise Randomization; Multiplicative Structural Nested Mean Model; Zero-inflated Outcome.

1. INTRODUCTION

Due to the ubiquitous presence of smartphones and the transition of casual gaming to mobile devices, mobile games are getting increasingly popular nowadays. As shown in a 2019 industry study by Golden Casino News, mobile games make up 60% of revenue for the global video game market, becoming the most significant segment of the video game industry. A common monetization strategy for mobile games is the freemium business model (Anderson, 2009), which provides free download and basic gameplay to attract customers, and then offers an option to pay for premium content such as in-game currency, extra content, or customization. The free part helps to increase the size of the user base quickly while the premium part generates revenue (Boudreau et al., 2019). It is shown that over 90% of mobile games begin as free, and over 90% of the profits from mobile games come from games that began as free (Banerjee et al., 2019). To retain users and stimulate consumption, developers offer promotions to players from time to time, such as sales on the add-on components, more in-game rewards, and holiday promotions. Understanding the effects of a sequence of promotion decisions on daily engagement, which usually refers to the amount of purchase, for heterogeneous users helps the game managers make improved/personalized promotion strategies.

Freemium mobile game data has its unique characteristics, which lead to several challenges in statistical inference. First, the outcomes, i.e., daily engagement, are zero-inflated. For active players, it is common to remain free users and never engage in the premium part of freemium games. Moreover, the positive daily engagement is skewed to the right. A 2014 study of freemium mobile games by Swrve found that 50% of mobile gaming revenue came from the top 10% of players making purchases, which only accounts for 0.15% of total players. Common distributional assumptions such as the normal or gamma are thereby no longer appropriate for freemium mobile game data.

Second, players are followed over a period of time, during which a sequence of treatments, promotion or no promotion, are implemented. We are interested in estimating the joint effects of a sequence of treatments on the outcomes rather than a one-time treatment. However, there exist time-varying confounders with the following characteristics: they are (i) associated with the outcomes, (ii) affected by earlier treatment, and (iii) predictive of subsequent treatments. For example, for estimating the promotion effects on user daily engagement, daily activity time is likely to be a time-varying confounder. This is because players with longer playing time are more likely to purchase in the game, players who received promotions tend to spend more time playing games, and activity time may also be an important factor in making subsequent promotion decisions. In the presence of time-varying confounders, standard regression methods, whether or not adjusting for time-varying confounders, are inappropriate for estimating the causal effects of a sequence of treatments (Robins & Hernán, 2009).

Lastly, in practice, the promotion assignment over time can be personalized or uniform. We refer to the treatment assignment that allocates the same treatment to all users at a given time as timewise randomization, while the personalized treatment assignment is individual randomization. In timewise randomization, since all users receive the same treatment at a given time, the convergence of the propensity score estimate solely relies on the number of time points rather than the sample size. In this case, the standard asymptotic framework requiring only the sample size to go to infinity is not sufficient to derive the large sample results, increasing the difficulties in estimation of treatment effects.

Existing works for estimating the causal effect of one-time treatment with semicontinuous outcomes with excessive zeros include two-part model (Duan et al., 1983), burden-of-illness model (Chang et al., 1994), and tobit model or its variants (Tobin, 1958; Powell, 1986; Keele & Miratrix, 2019; Cheng & Small, 2020). Estimating the causal effects of a sequence of treatments is considerably more challenging in the presence of time-varying confounding. Structural nested mean models (Robins, 1994) have been proposed to overcome this challenge by modeling treatment effects sequentially over time, and G-estimation can be used to isolate treatment effects in the presence of time-varying confounding. See Vansteelandt & Joffe (2014) for a review. However, existing structural nested mean models often require stringent model assumptions to handle zero-inflated outcomes. They require modeling (i) the probability of having positive outcomes and (ii) the conditional mean given a positive outcome separately, which amounts to specifying the treatment-specific conditional outcome means instead of the causal treatment effect functions directly. Moreover, the model parameter in part (ii) may not have a causal interpretation because it specifies the causal effect conditional on a post-treatment quantity. Besides, the standard asymptotic regime for G-estimation requires the sample size to increase to infinity and does not apply to the case when the number of follow-up times increases while the sample size can be fixed.

In this paper, we propose a class of multiplicative structural nested mean models for zero-inflated nonnegative outcomes. Our contributions are from several aspects. First, our proposed model describes flexibly and concisely the joint effects of a sequence of treatments in the presence of time-varying confounders based on the ratio of conditional means of outcomes, which can naturally accommodate zero-inflated nonnegative outcomes as demonstrated by the example given in the next section. Compared with the structural nested mean models which use the difference of conditional means as the contrast function, our model does not require the two potential outcome mean functions to be known. Second, for parameter estimation, we propose a class of doubly robust estimating equations, where the nuisance functions, i.e., propensity score and conditional means of outcomes given confounders, are estimated parametrically or nonparametrically. We show that the resulting estimator is rate doubly robust (Kallus & Mao, 2020; Farrell et al., 2021), that is, as long as the convergence rates of nonparametric estimators for nuisance functions are sufficiently fast, the resulting estimator is asymptotically normal and the typical sandwich formula can be used for consistent variance estimation without accounting for the variation due to the estimation of nuisance functions. Moreover, the conditional means of outcomes are estimated in two parts to leverage the characteristic of zero-inflated outcomes. That is, we separately model the probability of having positive outcomes and the mean outcome conditional on its being positive, both parts conditional on confounders. This facilitates the derivation of the rate doubly robust estimator with the desired theoretical properties. Third, in terms of theory development, we establish the consistency and asymptotic normality of the resulting rate doubly robust estimator as either the sample size or the follow-up time goes to infinity, i.e., bidirectional asymptotics (Shi et al., 2021), under the setting for individual randomization of treatment assignment. This task is nontrivial since individual data are dependent over time, and uniform consistency and weak convergence results need to be established for general martingale processes. We also establish similar results for the setting with timewise randomization as the follow-up time goes to infinity. However, the convergence rate is slower than the setting of individual randomization since it is only determined by the number of observations over time regardless of the sample size.

2. Review of Multiplicative Structural Nested Mean Model

The multiplicative structural nested mean model was proposed by Robins (1994) as an analogy to the structural nested mean model. In this paper, we utilize it as the treatment effect model for zero-inflated nonnegative outcomes. We start with a particular subject for simplicity. Suppose that measurements are collected at T discrete time points. Let L_t be the time-varying covariates collected at time point t, $A_t \in \{0, 1\}$ denote the treatment indicator (1 for promotion and 0 for no promotion) at time t, and Y_t stand for the observed outcome at time t $(t = 1, \ldots, T)$. We presume that the observed data are ordered as $L_1, A_1, Y_1, \ldots, L_T, A_T, Y_T$; thus the covariates and treatments precede the observed outcomes and Y_t can be a part of L_{t+1}. We use the overline notation to denote a variable's history, e.g., $\overline{A}_t = (A_1, \ldots, A_t)$. For notational convenience, let $V_t = (A_{t-1}, L_t)$ and $O_t = (L_t, A_t, Y_t)$, then the information available prior to treatment at time t is denoted by $\overline{V}_t = (\overline{A}_{t-1}, \overline{L}_t)$ and a subject's full record can be represented as $\overline{O}_T = \{(L_t, A_t, Y_t)\}_{1 \leq t \leq T}$.

We use the potential outcomes framework to define the causal effect of treatments. Let $Y_t^{(\overline{a}_T)}$ denote the potential outcome that would be seen at time t, had the subject received the sequence of treatments $\overline{a}_T$ through time T. In particular, $Y_t^{(\overline{0}_T)}$ is the potential outcome at time t had the subjects never received treatments. We are interested in estimating the causal effects of a sequence of treatments $\overline{a}_T$ on a group of users with covariates sequence $\overline{l}_T$, which can be defined as the ratio of the conditional expectation of the potential outcomes had this group of subjects received $\overline{a}_T$ and $\overline{0}_T$, i.e., $\mathbb{E}(Y_t^{(\overline{a}_T)} \mid \overline{A}_T = \overline{a}_T, \overline{L}_T = \overline{l}_T)/\mathbb{E}(Y_t^{(\overline{0}_T)} \mid \overline{A}_T = \overline{a}_T, \overline{L}_T = \overline{l}_T)$. Here we use the ratio rather than the difference of the conditional mean outcomes to accommodate zero-inflated nonnegative outcomes.

We presume that the treatments and covariates after time t cannot affect the potential outcomes at times up to t, i.e. $Y_t^{(\overline{a}_T)} = Y_t^{(\overline{a}_t)}$, and consider a class of multiplicative structural nested mean models in the following form:

$$\frac{\mathbb{E}(Y_t^{(\overline{a}_t)} \mid \overline{A}_T = \overline{a}_T, \overline{L}_T = \overline{l}_T)}{\mathbb{E}(Y_t^{(\overline{0}_t)} \mid \overline{A}_T = \overline{a}_T, \overline{L}_T = \overline{l}_T)} = \frac{\mathbb{E}(Y_t^{(\overline{a}_t)} \mid \overline{A}_t = \overline{a}_t, \overline{L}_t = \overline{l}_t)}{\mathbb{E}(Y_t^{(\overline{0}_t)} \mid \overline{A}_t = \overline{a}_t, \overline{L}_t = \overline{l}_t)} = \exp\{f_{\theta_0}(\overline{v}_t) a_t\} \; t = 1, \ldots, T, \quad (1)$$

where $\overline{v}_t = (\overline{l}_t, \overline{a}_{t-1})$ and $f_\theta(\cdot)$ is a known function with a p-dimensional vector of parameters $\theta \in \Theta \subseteq \mathbb{R}^p$ with true parameter value θ_0. Typically, the parameterization is chosen to be $f_\theta(\cdot) \equiv 0$ for $\theta = 0$, so that $\theta_0 = 0$ encodes no treatment effects. The proposed multiplicative structural nested mean model is semiparametric in nature because the conditional mean $\mathbb{E}(Y_t^{(\overline{0}_t)} \mid \overline{A}_t = \overline{a}_t, \overline{L}_t = \overline{l}_t)$ is completely unspecified.

Next, we use a toy example to illustrate the proposed treatment effect model.

Example 1. Use the freemium mobile game data as an illustration. Suppose users' potential daily purchase amounts follow a zero-inflated log-normal distribution. Specifically, $Y_t^{(\overline{a}_t)}$ has a probability p_t to be positive and follows a log-normal distribution *log-normal* (ν_t, σ_t^2) and a probability $(1 - p_t)$ to be zero, where p_t and ν_t are functions of the covariates and potential treatments up to time t. Suppose p_t follows a log-linear model and ν_t is a linear model. Then, the conditional mean of the potential outcome is

$$\begin{aligned}\mathbb{E}(Y_t^{(\overline{a}_t)} \mid \overline{A}_t = \overline{a}_t, \overline{L}_t = \overline{l}_t) &= p_t \exp(\nu_t + \frac{1}{2}\sigma_t^2) \\ &= \exp(\beta_p^T \tilde{l}_t + \gamma_p^T \tilde{l}_t a_t) \times \exp(\beta_\nu^T \tilde{l}_t + \gamma_\nu^T \tilde{l}_t a_t + \frac{1}{2}\sigma_t^2),\end{aligned}$$

where $\tilde{l}_t = (1, l_t^T)^T$ and $\beta_p, \gamma_p, \beta_\nu, \gamma_\nu$ are corresponding coefficients. Thus, the ratio of conditional means can be simplified as follows

$$\frac{\mathbb{E}(Y_t^{(\overline{a}_t)} \mid \overline{A}_t = \overline{a}_t, \overline{L}_t = \overline{l}_t)}{\mathbb{E}(Y_t^{(\overline{0}_t)} \mid \overline{A}_t = \overline{a}_t, \overline{L}_t = \overline{l}_t)} = \exp\{(\gamma_p + \gamma_\nu)^T \tilde{l}_t a_t\},$$

which satisfies model (1).

3. Main Methodology

3.1. *Identification and Estimation*

For the causal treatment effects to be estimable based on observed data, we require the following assumptions as widely used in the literature.

Assumption 1 (Consistency). The observed outcome is equal to the potential outcome under the sequence of actual treatments received; i.e., $Y_t = Y_t^{(\overline{A}_t)}$ $(t = 1, \ldots, T)$.

Assumption 2 (No unmeasured confounders). $A_t \perp\!\!\!\perp Y_t^{(\overline{0}_t)} \mid \overline{V}_t$ $(t = 1, \ldots, T)$, which means that A_t is conditionally independent of $Y_t^{(\overline{0}_t)}$ given $\overline{V}_t$.

The consistency assumption links the observed data to the potential outcome, which implicitly makes the stable unit treatment assumption that rules out multiple versions of treatment and interference.

We develop a G-estimator of the causal parameter θ_0 in model (1). Define

$$H_t(\theta_0) = Y_t \exp\left\{-f_{\theta_0}(\overline{V}_t) A_t\right\}.$$

Intuitively, $H_t(\theta_0)$ mimics the potential outcome $Y_t^{(\overline{0}_t)}$ that would have been seen had the treatment never been implemented. The following two lemmas establish the properties of $H_t(\theta_0)$.

LEMMA 1. *Under model* (1)*, Assumption 1 implies* $\mathbb{E}\{H_t(\theta_0) \mid A_t, \overline{V}_t\} = \mathbb{E}(Y_t^{(\overline{0}_t)} \mid A_t, \overline{V}_t), 1 \leq t \leq T$.

LEMMA 2. *Under model* (1)*, Assumptions 1 and 2 imply* $\mathbb{E}\{H_t(\theta_0) \mid A_t, \overline{V}_t\} = \mathbb{E}\{H_t(\theta_0) \mid \overline{V}_t\}, 1 \leq t \leq T$.

Let $h_0(\overline{v}_t) = \mathbb{E}\{H_t(\theta_0) \mid \overline{V}_t = \overline{v}_t\}$ and $\pi_0(\overline{v}_t) = P(A_t = 1 \mid \overline{V}_t = \overline{v}_t)$ denote the conditional mean of $H_t(\theta_0)$ and the propensity score, respectively. For any given propensity score π and conditional mean h, we construct a class of doubly robust estimating functions for θ:

$$\psi(\overline{O}_t; \theta, \pi, h, c) = c(\overline{V}_t)\left\{H_t(\theta) - h(\overline{V}_t)\right\}\left\{I(A_t = 1) - \pi(\overline{V}_t)\right\} \quad (t = 1, \ldots, T), \tag{2}$$

where $c(\cdot)$ is a p-dimensional function of the covariate and treatment history $\overline{V}_t$. It can be shown that under Lemma 2 the expectation of (2) is zero for $\theta = \theta_0$, regardless of c as long as $\pi = \pi_0$ or $h = h_0$. Suppose n subjects are randomly sampled from a population, we introduce another subscript for each variable to index subjects so that the observed data $\overline{O}_{1,T}, \dots, \overline{O}_{n,T}$ are independent and identically distributed copies of $\overline{O}_T$. The associated probability space is denoted by $(\Omega, \mathcal{F}, P)$. We consider the following estimating equation for θ:

$$G(\theta; \pi, h, c) \equiv \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \psi(\overline{O}_{i,t}; \theta, \pi, h, c) = 0.$$

The traditional definition of double robustness is applicable when the nuisance functions π_0, h_0 are estimated parametrically. A notion of rate double robustness is introduced for nonparametrically estimated nuisance functions, which describes the asymptotic normality of the resulting estimator and the consistent variance estimation of the typical sandwich formula when the convergence rates of nuisance function estimators are sufficiently fast (Kallus & Mao, 2020; Farrell et al., 2021). We will show the rate double robustness of the proposed estimator for θ_0 in § 3.2.

In practice, π_0 and h_0 are unknown and need to be estimated from data. The propensity score model can be estimated parametrically, such as with a logistic regression, or nonparametrically, such as with random forest. In mobile game data applications, a treatment is usually randomly assigned using either individual randomization or timewise randomization, where the propensity score can be estimated by the sample proportion. However, we cannot directly estimate h_0 by regressing $H_{i,t}(\theta_0)$ on $\overline{V}_{i,t}$ because $H_{i,t}(\theta_0)$ depends on the unknown parameters θ_0. Thus, we consider representing h_0 with other estimable conditional means. Define $\mu_{0,0}(\overline{v}_t) = \mathbb{E}(Y_t \mid A_t = 0, \overline{V}_t = \overline{v}_t)$ and $\mu_{1,0}(\overline{v}_t) = \mathbb{E}(Y_t \mid A_t = 1, \overline{V}_t = \overline{v}_t)$. Then, h_0 can be represented as

$$h_0(\overline{v}_t) = \mu_{1,0}(\overline{v}_t) \exp\{-f_{\theta_0}(\overline{v}_t)\}\pi_0(\overline{v}_t) + \mu_{0,0}(\overline{v}_t)\{1 - \pi_0(\overline{v}_t)\}.$$

Accordingly, we rewrite our estimating equation as

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \psi(\overline{O}_{i,t}; \theta, \hat{\pi}, \hat{\mu}_1, \hat{\mu}_0, c) = 0, \tag{3}$$

where $\hat{\pi}, \hat{\mu}_1, \hat{\mu}_0$ are the estimators of $\pi_0, \mu_{1,0}, \mu_{0,0}$ respectively, and

$$\begin{aligned}\psi(\overline{O}_{i,t}; \theta, \hat{\pi}, \hat{\mu}_1, \hat{\mu}_0, c) = c(\overline{V}_{i,t}) \big[H_{i,t}(\theta) - \hat{\mu}_1(\overline{V}_{i,t}) \exp\{-f_\theta(\overline{V}_{i,t})\}\hat{\pi}(\overline{V}_{i,t}) \\ - \hat{\mu}_0(\overline{V}_{i,t})\{1 - \hat{\pi}(\overline{V}_{i,t})\}\big] \times \big\{I(A_{i,t} = 1) - \hat{\pi}(\overline{V}_{i,t})\big\}.\end{aligned}$$

Taking into account that outcomes are zero-inflated, $\mu_{1,0}$ and $\mu_{0,0}$ can be represented as

$$\mu_{a,0}(\overline{v}_t) = \mathbb{P}(Y_t > 0 \mid A_t = a, \overline{V}_t = \overline{v}_t) \times \mathbb{E}(Y_t \mid Y_t > 0, A_t = a, \overline{V}_t = \overline{v}_t) \quad (a = 0, 1). \tag{4}$$

To improve the estimation accuracy, we estimate $\mu_{1,0}$ and $\mu_{0,0}$ by modeling the probability part and the mean part on the right-hand side of (4) separately, using some nonparametric regression techniques, such as generalized additive models. We show the advantage of the two-part estimation method of conditional outcome means compared with the one-part approach in terms of accuracy and efficiency in § B.1 of the supplementary material. The two-part method is favorable when zero values account for a large proportion of outcomes and the sample size is small.

The choice of function $c(\cdot)$ generally does not affect the consistency of the estimator but may make a difference in the efficiency. Following Robins (1994) and Lok (2021), it can be shown that under certain conditions, an efficient estimator of θ_0 can be obtained by setting

$$c(\overline{v}_t) = \left. \frac{\mathbb{E}\left(\frac{\partial H_t(\theta)}{\partial \theta} \mid A_t = 1, \overline{V}_t = \overline{v}_t\right) - \mathbb{E}\left(\frac{\partial H_t(\theta)}{\partial \theta} \mid A_t = 0, \overline{V}_t = \overline{v}_t\right)}{\operatorname{var}(Y_t^{(\overline{0}_t)} \mid \overline{V}_t = \overline{v}_t)} \right|_{\theta = \theta_0}. \tag{5}$$

The optimal c function depends on the conditional variance of $Y_t^{(\overline{0}_t)}$, which may require an additional working model and is difficult to estimate well based on observed data. The estimation of it also causes efficiency loss of the proposed estimator. In practice, we can choose a simple function for c, such as

$c(\overline{v}_t) = \partial f_\theta(\overline{v}_t)/\partial\theta$. We compare the empirical performance of the estimators constructed based on simple c and optimal c in our simulations; see § 4.

The proposed estimating equation (3) can be solved using the standard Newton-Raphson method. One possible challenge is that the estimating equation may not converge if the initial value is not chosen appropriately. A common solution is to try different initial values and choose one that can lead to a proper solution.

3.2. *Asymptotic Distribution and Variance Estimation*

Let $\hat{\theta}$ denote the proposed estimator by solving the estimating equation (3). Before delving into the theoretical analysis of $\hat{\theta}$, we introduce more notation for simplicity of exposition. Define the triplet function $\eta = (\pi, \mu_1, \mu_0)$. The true value of η is $\eta_0 = (\pi_0, \mu_{1,0}, \mu_{0,0})$, and its estimator is $\hat{\eta} = (\hat{\pi}, \hat{\mu}_1, \hat{\mu}_0)$. Let $\psi_{i,t}(\theta, \eta) = \psi(\overline{O}_{i,t}; \theta, \pi, \mu_1, \mu_0, c)$ and

$$\mathbb{P}g\{\psi(\theta, \eta)\} = \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[g\{\psi_{1,t}(\theta, \eta)\}\right], \tag{6}$$

$$\mathbb{P}_n g\{\psi(\theta, \eta)\} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} g\{\psi_{i,t}(\theta, \eta)\}, \tag{7}$$

where g is any given function or operator of $\psi_{i,t}$, e.g., $g\{\psi_{i,t}(\theta, \eta)\} = \partial\psi_{i,t}(\theta, \eta)/\partial\theta$. In formula (6), we assume $T \in \mathbb{N}^+ \cup \{\infty\}$ and define $T^{-1}\sum_{t=1}^{T} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}$ when $T = \infty$. Denote the Euclidean norm by $\|\cdot\|_2$ and the $\mathcal{L}_2(P)$ norm by $\|\cdot\|_{2,P}$ with the definition $\|\eta\|_{2,P}^2 = T^{-1}\sum_{t=1}^{T}\int \|\eta(\overline{V}_t)\|_2^2 \, dP(\overline{V}_t)$. Define a function set $\mathcal{G}_{\eta_0} = \{\eta : ||\eta - \eta_0||_{2,P} < \delta\}$ for some $\delta > 0$ and a Cartesian product $\mathcal{U} = \{(\theta, \eta) : \theta \in \Theta, \eta \in \mathcal{G}_{\eta_0}\}$.

To establish the asymptotic normality of $\hat{\theta}$ under individual randomization, we require the following conditions.

Condition 1. The solution to $\mathbb{P}\psi(\theta, \eta_0) = 0$ is unique. $\|\mathbb{P}\psi(\theta_n, \eta_0)\|_2 \to 0$ implies $\|\theta_n - \theta_0\|_2 \to 0$ for any sequence of $\{\theta_n\} \in \Theta$.

Condition 2. There exists a finite ϵ-net $\mathcal{U}_\epsilon$ of $\mathcal{U}$ for any $\epsilon > 0$. In addition, $\mathcal{G}_{\eta_0}$ has uniformly integrable entropy. That is, $\int_0^\infty \sup_Q \sqrt{\log N(\epsilon\|F\|_{2,Q}, \mathcal{G}_{\eta_0}, \|\cdot\|_{2,Q})}\, d\epsilon < \infty$, where $F : \Omega \to \mathbb{R}^3$ is a square integrable envelop for $\mathcal{G}_{\eta_0}$, and the covering number $N(\epsilon, \mathcal{G}_{\eta_0}, \|\cdot\|)$ is the minimum number of balls $\{\eta' : \|\eta' - \eta\| < \epsilon\}$ of radius ϵ needed to cover $\mathcal{G}_{\eta_0}$.

Condition 3. (i) $|f_\theta(\overline{V}_t)|$ $(t = 1, \ldots, T)$ is bounded almost surely for all $\theta \in \Theta$. (ii) $|Y_t|$, $\|c(\overline{V}_t)\|_2$, $\sigma_{0,0}^2(\overline{V}_t)$ and $\sigma_{1,0}^2(\overline{V}_t)$ $(t = 1, \ldots, T)$ are bounded almost surely, where $\sigma_{a,0}^2(\overline{v}_t) = \mathrm{var}(Y_t \mid A_t = a, \overline{V}_t = \overline{v}_t)$ $(a = 0, 1)$. (iii) $\|\eta(\overline{V}_t)\|_2 < b_0$ $(t = 1, \ldots, T)$ almost surely for all $\eta \in \mathcal{G}_{\eta_0}$ for some constant b_0. (iv) $\|\partial\psi_{1,t}(\theta, \eta)/\partial\eta^T\|_2 = \|(\partial\psi_{1,t}(\theta, \eta)/\partial\pi, \partial\psi_{1,t}(\theta, \eta)/\partial\mu_1, \partial\psi_{1,t}(\theta, \eta)/\partial\mu_0)\|_2 < b^*$ $(t = 1, \ldots, T)$ almost surely for all $\theta \in \Theta$, $\eta \in \mathcal{G}_{\eta_0}$ for some constant b^*.

Condition 4. As $nT \to \infty$, $\|\hat{\eta} - \eta_0\|_{2,P} \xrightarrow{P} 0$, and

$$(\|\hat{\mu}_1 - \mu_{1,0}\|_{2,P} + \|\hat{\mu}_0 - \mu_{0,0}\|_{2,P} + \|\hat{\pi} - \pi_0\|_{2,P})\|\hat{\pi} - \pi_0\|_{2,P} = o_P\{(nT)^{-1/2}\}.$$

Condition 5. For each $\eta \in \mathcal{G}_{\eta_0}$,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \mathbb{E}\left\{m_{i,t}(\eta)m_{i,t}^T(\eta) \mid \overline{V}_{i,t}\right\} \xrightarrow{P} \Sigma(\eta) \text{ as } nT \to \infty, \tag{8}$$

where $m_{i,t}(\eta) = \psi_{i,t}(\theta_0, \eta) - \mathbb{E}\left\{\psi_{i,t}(\theta_0, \eta) \mid \overline{V}_{i,t}\right\}$ and $\Sigma(\eta)$ is a constant positive definite matrix for each η.

Condition 6. For $j = 1, \ldots, p$, there exists a constant M such that as $nT \to \infty$

$$P\left(\sup_{\eta,\eta' \in \mathcal{G}_{\eta_0}} \frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\mathbb{E}\left[\left\{m_{i,t}^{(j)}(\eta) - m_{i,t}^{(j)}(\eta')\right\}^2 \mid \overline{V}_{i,t}\right]}{\|\eta - \eta'\|_{2,P}^2} \geq M\right) \to 0,$$

where $m_{i,t}(\eta)$ is defined in Condition 5 and $m_{i,t}^{(j)}(\eta)$ is the jth component of $m_{i,t}(\eta)$.

Condition 1 is the identifiability condition, which is commonly assumed to ensure the consistency of Z-estimators (see Theorem 2.10 of Kosorok (2007)). Condition 2 controls the complexity of the function set $\mathcal{G}_{\eta_0}$ to show the uniform convergence (Lemma A1) and weak convergence (Lemma A2) of a function-indexed martingale difference sequence. Similar conditions are used in Theorem 2.3 of Kosorok (2007) and Theorem 2 of Bae et al. (2010). In fact, many important classes of functions, such as VC graph classes, have uniformly integrable entropy. See § 2.6 of Van Der Vaart & Wellner (1996) for details. Condition 3 is standard, which generally holds when the covariates L, outcomes Y, and parameters are bounded.

Condition 4 provides the required convergence rates for the estimators of the nuisance functions under individual randomization. The first part helps to establish the consistency of the estimator $\hat{\theta}$, while the second part is used to derive the asymptotic distribution of $\hat{\theta}$. This condition is widely used in the causal inference literature to derive the asymptotic distribution of doubly robust estimators when nuisance functions are estimated parametrically or nonparametrically with proper rates (see e.g. Kallus & Mao, 2020; Farrell et al., 2021). For example, if π is estimated based on a correctly specified parametric model so that $\|\hat{\pi} - \pi_0\|_{2,P} = O_P\{(nT)^{-1/2}\}$, then we only need $\hat{\mu}_0$ and $\hat{\mu}_1$ to be consistentOn the other hand, if all the nuisance functions are estimated nonparametrically, we require $\|\hat{\pi} - \pi_0\|_{2,P} = o_P\{(nT)^{-1/4}\}$, $\|\hat{\mu}_0 - \mu_{0,0}\|_{2,P} = o_P\{(nT)^{-1/4}\}$, and $\|\hat{\mu}_1 - \mu_{1,0}\|_{2,P} = o_P\{(nT)^{-1/4}\}$. Many nonparametric/semiparametric estimators can satisfy the convergence rate of $o_P\{(nT)^{-1/4}\}$, such as single-index models, generalized additive models, and partially linear models (Horowitz, 2009).

Conditions 5 and 6 are imposed to derive the weak convergence (Lemma A2). Their validity is discussed in Remarks 1 and 2 below, respectively.

Remark 1 (Discussion of Condition 5). The left-hand side of (8) can be calculated as

$$\begin{aligned}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\Big\{c(\overline{V}_{i,t})c^T(\overline{V}_{i,t})\Big(&\{1-\pi(\overline{V}_{i,t})\}^2\pi_0(\overline{V}_{i,t})\exp\{-2f_{\theta_0}(\overline{V}_{i,t})\}\sigma_{1,0}^2(\overline{V}_{i,t})\\ &+\ \pi^2(\overline{V}_{i,t})\{1-\pi_0(\overline{V}_{i,t})\}\sigma_{0,0}^2(\overline{V}_{i,t}) + \pi_0(\overline{V}_{i,t})\{1-\pi_0(\overline{V}_{i,t})\}\\ &\times\ \big[\mu_{1,0}(\overline{V}_{i,t})\exp\{-f_{\theta_0}(\overline{V}_{i,t})\}\pi_0(\overline{V}_{i,t}) + \mu_{0,0}(\overline{V}_{i,t})\{1-\pi_0(\overline{V}_{i,t})\}\\ &-\ \mu_1(\overline{V}_{i,t})\exp\{-f_{\theta_0}(\overline{V}_{i,t})\}\pi(\overline{V}_{i,t}) + \mu_0(\overline{V}_{i,t})\{1-\pi(\overline{V}_{i,t})\}\big]^2\Big)\Big\}.\end{aligned}$$

When $n \to \infty$ and T is finite, Condition 5 holds under mild conditions by applying the *law of large numbers*. When $T \to \infty$ and n is finite, letting

$$X_t = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\{m_{i,t}(\eta)m_{i,t}^T(\eta) \mid \overline{V}_{i,t}\right\} \quad (t = 1, \ldots, T),$$

$\{X_t\}_{t\geq 1}$ forms a stochastic process. If $\{X_t\}_{t\geq 1}$ is uniformly integrable and α-mixing, by Theorem 1 of Andrews (1988), we have $T^{-1}\sum_{t=1}^{T}X_t \xrightarrow{P} T^{-1}\sum_{t=1}^{T}\mathbb{E}(X_t)$. Thus, Condition 5 also holds under this setting.

Remark 2 (Discussion of Condition 6). Let $\eta_{i,t} = \eta(\overline{V}_{i,t})$. Then, we have

$$\mathbb{E}\left[\left\{m_{i,t}^{(j)}(\eta_{i,t}) - m_{i,t}^{(j)}(\eta'_{i,t})\right\}^2 \mid \overline{V}_{i,t}\right]$$

$$= \mathbb{E}\left[\left\{\frac{\partial m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}^T \left(\eta_{i,t}-\eta'_{i,t}\right)\left(\eta_{i,t}-\eta'_{i,t}\right)^T \left\{\frac{\partial m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\} \mid \overline{V}_{i,t}\right]$$
$$= \mathbb{E}\left[\left(\eta_{i,t}-\eta'_{i,t}\right)^T \left\{\frac{\partial m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}\left\{\frac{\partial m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}^T \left(\eta_{i,t}-\eta'_{i,t}\right) \mid \overline{V}_{i,t}\right]$$
$$= \left(\eta_{i,t}-\eta'_{i,t}\right)^T \mathbb{E}\left[\left\{\frac{\partial m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}\left\{\frac{\partial m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}^T \mid \overline{V}_{i,t}\right]\left(\eta_{i,t}-\eta'_{i,t}\right),$$

where $\|\tilde{\eta}-\eta\|_2 < \|\eta'-\eta\|_2$. If the largest eigenvalue of the expectation term in the above quadratic form is uniformly bounded over i, t by some constant M, we can get

$$\sup_{\eta,\eta'\in\mathcal{G}_{\eta_0}} \frac{\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T \mathbb{E}\left[\left\{m_{i,t}^{(j)}(\eta_{i,t})-m_{i,t}^{(j)}(\eta'_{i,t})\right\}^2 \mid \overline{V}_{i,t}\right]}{\|\eta-\eta'\|_{2,P}^2}$$
$$= \sup_{\eta,\eta'\in\mathcal{G}_{\eta_0}} \frac{\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T \left(\eta_{i,t}-\eta'_{i,t}\right)^T \mathbb{E}\left[\left\{\frac{m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}\left\{\frac{m_{i,t}^{(j)}(\eta_{i,t})}{\partial \eta_{i,t}}|_{\eta=\tilde{\eta}}\right\}^T \mid \overline{V}_{i,t}\right]\left(\eta_{i,t}-\eta'_{i,t}\right)}{\|\eta-\eta'\|_{2,P}^2}$$
$$\leq M \sup_{\eta,\eta'\in\mathcal{G}_{\eta_0}} \frac{\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T \|\eta_{i,t}-\eta'_{i,t}\|_2^2}{\|\eta-\eta'\|_{2,P}^2}.$$

If we can show $(nT)^{-1}\sum_{i=1}^n\sum_{t=1}^T \|\eta_{i,t}-\eta'_{i,t}\|_2^2 \xrightarrow{P} \|\eta-\eta'\|_{2,P}^2$ as $nT\to\infty$, we are able to derive Condition 6. The proof of the above convergence can be similarly derived as given in Remark 1.

In timewise randomization, because the convergence rate of the estimator for propensity score depends only on the length of follow-up time, we replace Condition 4 with Condition 7 below.

Condition 7. As $T\to\infty$, $\|\hat{\eta}-\eta_0\|_{2,P} \xrightarrow{P} 0$, and

$$(\|\hat{\mu}_1-\mu_{1,0}\|_{2,P}+\|\hat{\mu}_0-\mu_{0,0}\|_{2,P}+\|\hat{\pi}-\pi_0\|_{2,P})\|\hat{\pi}-\pi_0\|_{2,P} = o_P(T^{-1/2}).$$

Conditions 5 and 6 also need to be modified. Specifically, define $\tilde{m}_t(\eta) = n^{-1}\sum_{i=1}^n \psi_{i,t}(\theta_0,\eta) - \mathbb{E}\left\{n^{-1}\sum_{i=1}^n \psi_{i,t}(\theta_0,\eta) \mid \overline{V}_{\cdot,t}\right\}$ $(t=1,\dots,T)$, where $\overline{V}_{\cdot,t} = (\overline{V}_{1,t},\dots,\overline{V}_{n,t})$. We require the following modified conditions.

Condition 8. For each $\eta\in\mathcal{G}_{\eta_0}$, $T^{-1}\sum_{t=1}^T \mathbb{E}\left\{\tilde{m}_t(\eta)\tilde{m}_t^T(\eta) \mid \overline{V}_{\cdot,t}\right\} \xrightarrow{P} \Sigma_1(\eta)$ as $T\to\infty$, where $\Sigma_1(\eta)$ is a constant positive definite matrix for each η.

Condition 9. For $j=1,\dots,p$, there exists a constant M such that as $T\to\infty$

$$P\left(\sup_{\eta,\eta'\in\mathcal{G}_{\eta_0}} \frac{\frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[\left\{\tilde{m}_t^{(j)}(\eta)-\tilde{m}_t^{(j)}(\eta')\right\}^2 \mid \overline{V}_{\cdot,t}\right]}{\|\eta-\eta'\|_{2,P}^2} \geq M\right) \to 0,$$

where $\tilde{m}_t^{(j)}(\eta)$ is the jth component of $\tilde{m}_t(\eta)$.

We establish the bidirectional asymptotics of $\hat{\theta}$ under individual randomization in Theorem 1 and the asymptotic distribution of $\hat{\theta}$ under timewise randomization in Theorem 2.

THEOREM 1 (BIDIRECTIONAL ASYMPTOTICS). *If the treatment assignment is individual randomization, under model (1) and Conditions 1–6, as either $n\to\infty$ or $T\to\infty$, we have* $\sqrt{nT}(\hat{\theta}-\theta_0) \xrightarrow{d}$

$MVN\left(0, B\Sigma B^T\right)$, where $B = \left\{\mathbb{P}\dot{\psi}_\theta(\theta_0, \eta_0)\right\}^{-1}$, $\dot{\psi}_\theta(\theta, \eta) = \partial\psi(\theta, \eta)/\partial\theta^T$, and

$$\Sigma = \Sigma(\eta_0) = \lim_{nT\to\infty} \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \mathbb{E}\left\{\psi_{i,t}(\theta_0, \eta_0)\psi_{i,t}^T(\theta_0, \eta_0) \mid \overline{V}_{i,t}\right\}.$$

In addition, B and Σ can be estimated by

$$\hat{B} = \left\{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \frac{\partial\psi_{i,t}(\theta, \hat{\eta})}{\partial\theta^T}\Big|_{\theta=\hat{\theta}}\right\}^{-1}, \quad \hat{\Sigma} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\psi_{i,t}(\hat{\theta}, \hat{\eta})\psi_{i,t}^T(\hat{\theta}, \hat{\eta}).$$

THEOREM 2. *If the treatment assignment is timewise randomization, under model (1) and Conditions 1–3 and 7–9, as $T \to \infty$, we have $\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} MVN\left(0, B\Sigma_1 B^T\right)$, where B is defined as the same as in Theorem 1 and*

$$\Sigma_1 = \Sigma_1(\eta_0) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\left\{\frac{1}{n}\sum_{i=1}^{n}\psi_{i,t}(\theta_0, \eta_0)\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}\psi_{i,t}(\theta_0, \eta_0)\right\}^T \mid \overline{V}_{\cdot,t}\right].$$

In addition, Σ_1 can be estimated by

$$\hat{\Sigma}_1 = \frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{n}\sum_{i=1}^{n}\psi_{i,t}(\hat{\theta}, \hat{\eta})\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}\psi_{i,t}(\hat{\theta}, \hat{\eta})\right\}^T.$$

The proofs of Theorems 1 and 2 are given in the Supplementary Material. Both theorems state that the asymptotic variance of the proposed estimator $\hat{\theta}$ can be consistently estimated by the typical sandwich formula without accounting for the variation of $\hat{\eta}$. This is mainly due to the rate double robustness for $\hat{\eta}$ that satisfies Condition 4 or 7.

4. SIMULATION STUDY

We conduct Monte Carlo simulations to examine the finite sample performance of the proposed estimator. We consider three covariates, which are generated as below:

$$L_{i,t}^{(1)} = \log(t),\ L_{i,t}^{(2)} = \log(K_{i,t}^{(2)} + 1),\ L_{i,t}^{(3)} = L_t^{(3)}\ (i = 1, \ldots, n;\ t = 1, \ldots, T),$$

where $K_{i,t}^{(2)} \sim (1 - q_{i,t})I(K_{i,t}^{(2)} = 0) + q_{i,t}\,\text{log-normal}\left(w_{i,t}, 0.5^2\right)$ and $L_t^{(3)} \sim \text{Ber}\,(0.5)$, with $q_{i,t} = \{1 + \exp(1.1 - 0.5 l_{i,t-1}^{(2)} - l_{i,t}^{(3)} - 0.5 a_{i,t-1})\}^{-1}$ and $w_{i,t} = 0.8 l_{i,t-1}^{(2)} + 0.6 l_{i,t}^{(3)} + 0.25 a_{i,t-1}$. The first covariate is to model the time trend effect of treatment; the second covariate is a log transformation of $K_{i,t}$, which mimics the daily activity time in mobile game data and follows a zero-inflated log-normal distribution; the third covariate gives the daily effect of treatment, such as weekday vs. weekend.

The treatment $A_{i,t}$ is generated from a Bernoulli distribution by two means: (i) individual randomization; (ii) timewise randomization, i.e., $A_{i,t} = A_{j,t}$ for any i, j. In the first scenario, we consider a constant propensity score $\pi_0 = 0.5$ and a covariates-dependent propensity score $\pi_0(\overline{v}_{i,t}) = \{1 + \exp(0.5 - 0.5 l_{i,t}^{(2)} - 0.8 l_{i,t}^{(3)})\}^{-1}$. We consider various sample sizes n with a fixed follow-up time T. In the second scenario, we only consider a constant propensity score $\pi_0 = 0.5$, with a fixed n and increasing T.

We generate the potential outcomes $Y_{i,t}^{(\overline{a}_{i,t})}$ from a zero-inflated log-normal distribution

$$(1 - p_{i,t}^{(\overline{a}_{i,t})})I(Y_{i,t}^{(\overline{a}_{i,t})} = 0) + p_{i,t}^{(\overline{a}_{i,t})}\,\text{log-normal}\,(\nu_{i,t}^{(\overline{a}_{i,t})}, 0.5^2),$$

where $p_{i,t}^{(\overline{a}_{i,t})} = \exp(\beta_p^T\tilde{l}_{i,t} + \gamma_p^T\tilde{l}_{i,t}a_{i,t})$ and $\nu_{i,t}^{(\overline{a}_{i,t})} = \beta_\nu^T\tilde{l}_{i,t} + \gamma_\nu^T\tilde{l}_{i,t}a_{i,t}$ with $\tilde{l}_{i,t} = (1, l_{i,t}^T)^T$. Here, we choose small β_p, γ_p to ensure $p_{i,t}^{(\overline{a}_{i,t})} < 1$. Then, the true $f_{\theta_0}(\cdot)$ in model (1) is $f_{\theta_0}(\overline{v}_{i,t}) = \theta_0^T\tilde{l}_{i,t}$ with $\theta_0 = \gamma_p + \gamma_\nu$. In practice, the treatment may have opposite effects on the proportion of users with positive outcomes and the conditional mean of positive outcomes. For example, sales on the add-on components in the game may increase the proportion of players purchasing them, but it also decreases the average cost for the users who pay for them. In our simulation, we consider two scenarios: (i) the treatment effect has the same direction on the probability of having positive outcomes and the conditional mean of positive outcomes, i.e., $\gamma_p \geq 0, \gamma_\nu \geq 0$ or $\gamma_p \leq 0, \gamma_\nu \leq 0$, where $\geq$ or $\leq$ stands for the point-wise comparison; (ii) the treatment effect has the opposite direction on the probability of having positive outcomes and the conditional mean of positive outcomes, i.e., $\gamma_p \geq 0, \gamma_\nu \leq 0$ or $\gamma_p \leq 0, \gamma_\nu \geq 0$. The settings are described as below:

$$\text{same direction: } \beta_p = (-0.8, -0.3, 0, 0)^T, \beta_\nu = (0, 0, 0.3, 0.5)^T, \gamma_p = (0.2, 0.15, 0, 0)^T,$$
$$\gamma_\nu = (0.1, 0, 0.05, 0.08)^T, \theta_0 = (0.3, 0.15, 0.05, 0.08)^T;$$
$$\text{opposite direction: } \beta_p = (-0.6, -0.15, 0, 0)^T, \beta_\nu = (0, 0, 0.3, 0.5)^T, \gamma_p = (-0.2, -0.15, 0, 0)^T,$$
$$\gamma_\nu = (0.1, 0, 0.05, 0.08)^T, \theta_0 = (-0.1, -0.15, 0.05, 0.08)^T.$$

The observed outcome is $Y_{i,t} = Y_{i,t}^{(\overline{A}_{i,t})}$ $(i = 1, \ldots, n;\ t = 1, \ldots, T)$.

The parameters θ_0 are estimated by solving the estimating equation (3), where the propensity score π_0 is nonparametrically estimated by fitting a generalized additive model for covariates-dependent propensity score and by sample proportion for constant propensity score. The conditional means of outcomes μ_0 and μ_1 are estimated nonparametrically by building a generalized additive model on the probability of being positive and the conditional mean for positive outcomes, respectively. In addition, we consider two choices of function c in the estimating equation: an optimal function $c(\overline{v}_{i,t})$ in (5) computed based on the true model, denoted by Opt. c, and a simple function $c(\overline{v}_{i,t}) = (1, l_{i,t}^{(1)}, l_{i,t}^{(2)}, l_{i,t}^{(3)})^T$, denoted by Sim. c.

For each simulation setting, we conduct 1000 replicates and report the average (Mean) and standard deviation (SD) of the estimates, the average of the estimated standard error (SE) of the estimator using the sandwich formula given in Theorems 1 and 2, and the empirical coverage probability (C.P.%) of the 95% Wald-type confidence interval. Due to limited space, we only show point estimation results in the main paper by boxplot, and leave tables with variance estimation results in § B.3 of the supplementary material. Fig. 1 and 2 (Tables A5 and A6 in § B.3) respectively show the results for the settings with the same and opposite direction of treatment effects under individual randomization. Based on the results, we can see that the proposed estimators are nearly unbiased, the average of estimated standard errors is close to the standard deviation of the estimates, and the empirical coverage probability is close to the nominal level. In addition, the estimators obtained using the simple c function show comparable performance compared with those obtained using the optimal c function under our considered settings, with slightly larger standard deviations. Note that in our simulations the optimal c is computed based on the true model. In practice, we should take into account the efficiency loss due to the estimation of the optimal c. We also considered cases with a fixed $n = 100$ and increasing T from 60 to 1000. The results are similar, which are omitted here.

Next, we consider settings with timewise randomization. Fig. 3 (Tables A7 and A8 in § B.3) shows the results under timewise randomization. We observe that the proposed estimator is nearly unbiased even with $T = 60$. However, when $T = 60$, the average of the estimated standard errors is smaller than the standard deviation of the estimators for some parameters, like intercept and time trend effect. As such, the empirical coverage probability is lower than the nominal level for these parameters. But as T increases, the average of the estimated standard errors gets much closer to the standard deviation of estimates and the resulting empirical coverage probability is close to the nominal level. The estimators obtained using the simple c function and optimal c function show comparable efficiency as in the individual randomization settings. These results demonstrate the validity of our inference procedure for both individual randomization and timewise randomization settings.

In the simulation, we have showed the rate double robustness of the proposed estimator where both conditional means and propensity score are estimated semiparametrically or nonparametrically. The traditional double robustness does not apply to our method since the estimation of h_0 depends on the estimation of the propensity score and h_0 is misspecified when the propensity score is misspecified. Nonetheless, the

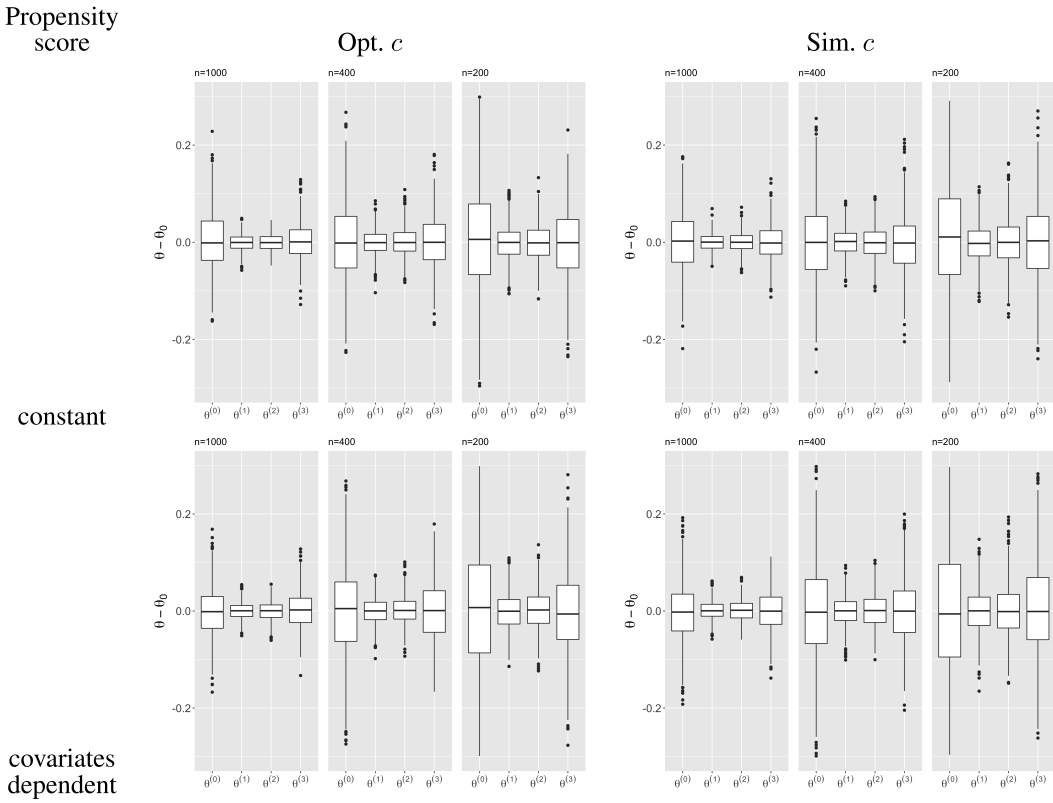


Fig. 1: Boxplots of estimators with same directional treatment effect and $T = 60$ under individual randomization. Left plots and right plots have different choices of function c. Top plots and bottom plots have different types of propensity scores.

other side of the traditional double robustness is shown by simulations in § B.1 of the supplementary material, where the conditional outcome means are estimated using misspecified parametric models while the propensity score is fitted nonparametrically.

5. Application to Freemium Mobile Game Data

We apply our method to a real dataset from a freemium mobile game (Banerjee et al., 2019), where players fight each other to earn rewards and can purchase items to improve their chances to win and/or the general user experience. Apart from the direct in-app purchase, there are several indirect ways of monetizing freemium games by engaging their free users in promoting the games through social media, for example, inviting friends on Facebook. As in Banerjee et al. (2019), the daily engagement of a player is defined as the combination of his/her in-app purchase (direct) and varied involvement with the game on social media (indirect) during the day. Daily activity is another metric of interest in mobile games, which measures the time a player spends playing the game during the day. Generally speaking, positive daily activity does not always imply positive daily engagement, but positive and high daily engagement often follows the persistent and incremental positive daily activity (Banerjee et al., 2019).

There are 38,860 players in the dataset whose daily engagement and daily activity were tracked for 60 consecutive days starting from the release date of the game. Approximately 51.8% of the players have positive daily engagement (direct or indirect) during the follow-up period. All players received the same history of promotion decisions (timewise randomization), which alternatively include 40 days of promotions and 20 days of no promotions. We want to model the effect of a sequence of promotion

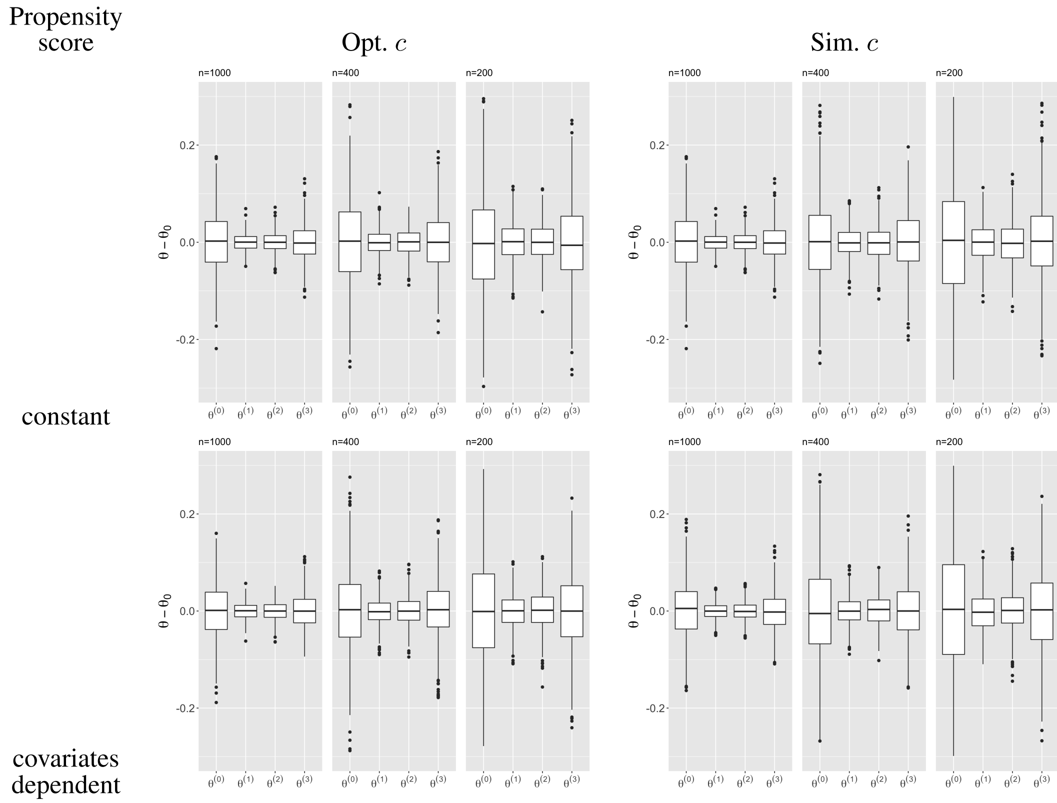


Fig. 2: Boxplots of estimators with opposite directional treatment effect and $T = 60$ under individual randomization. Left plots and right plots have different choices of function c. Top plots and bottom plots have different types of propensity scores.

decisions on the daily engagement in the presence of other time-varying variables, for example, daily activity.

We consider three covariates at time t: the number of days t in the study, the daily activity on the previous day $l_{i,t}^{(2)}$, and the weekend indicator $l_t^{(3)}$. We fit model (1) with the function $f_\theta(\cdot)$ given by

$$f_{\theta_0}(\overline{v}_{i,t}) = \theta_0^{(0)} + \theta_0^{(1)} \log t + \theta_0^{(2)} \log(l_{i,t}^{(2)} + 1) + \theta_0^{(3)} l_t^{(3)}.$$

Our proposed estimator of θ_0 is $\hat{\theta} = (-0.77, 0.50, -0.26, 0.05)^T$, which is obtained based on the simple c function. Its standard error estimate is $(0.121, 0.057, 0.031, 0.166)^T$, which is obtained using the proposed sandwich formula in Theorem 2. Based on the results, we make a few observations. First, with a significance level of 0.05, promotions have a significant positive interaction effect with the log of time on daily engagement. In general, users may lose interest in the game over time. This implies that promotions over time are able to provide incentives and grab their attention again. Second, promotions have smaller effects in terms of increasing the engagement for active users (i.e., those with large daily activity) than for less active users, because the estimator of $\theta_0^{(2)}$ is negative with a p-value less than 0.05. This finding is coherent with the literature since active users tend to be less sensitive to the promotion compared with less active users in terms of engagement. Last, there is no significant difference between the promotion effects on weekends and weekdays with a significance level of 0.05.

To visualize the treatment effects, we plot the estimated log-ratio treatment effect $f_{\hat{\theta}}(\overline{v}_{i,t})$ as a function of time t and log of daily activity plus 1 $\log(l_{i,t}^{(2)} + 1)$, stratified by the weekend vs. weekdays (Fig. 4a and Fig. 4b). Here, $f_{\hat{\theta}}(\overline{v}_{i,t}) > 0$ indicates a positive treatment effect of promotions on daily engagement, while

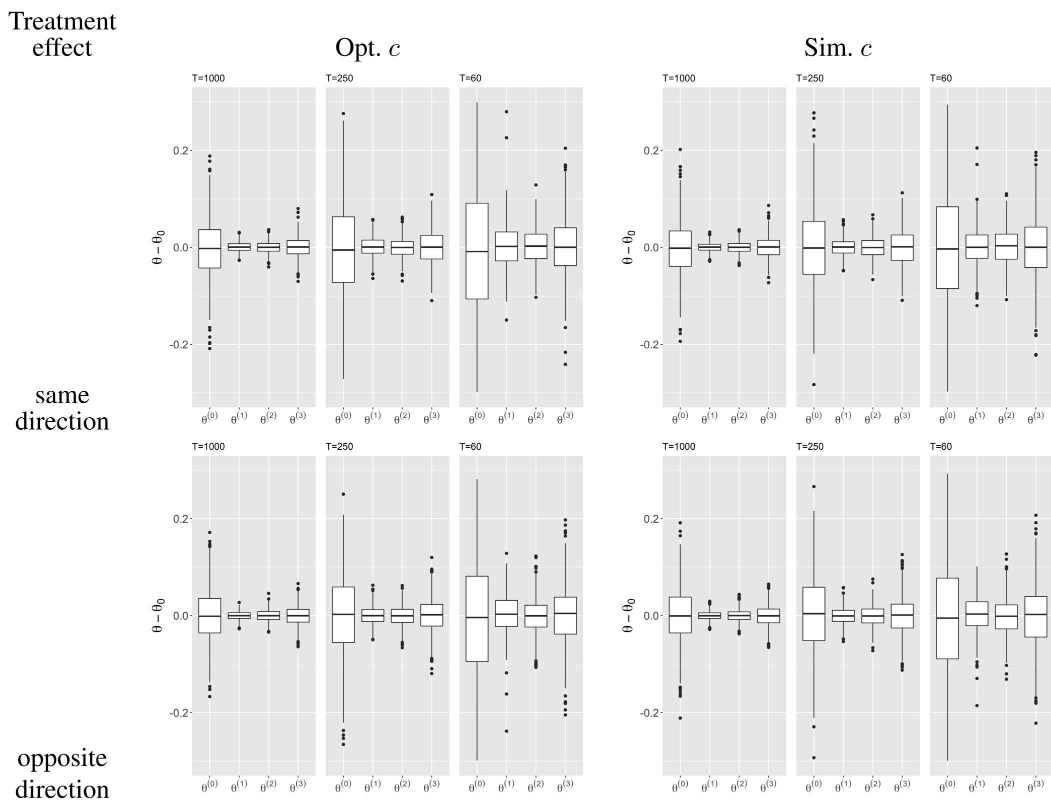


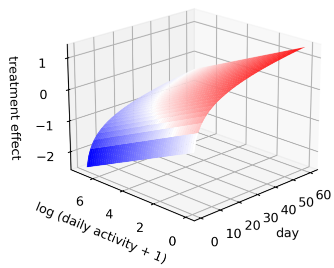
Fig. 3: Boxplots of estimators with $n = 400$ under timewise randomization. Left plots and right plots have different choices of function c. Top plots and bottom plots have different directions of treatment effects.

$f_{\hat{\theta}}(\overline{v}_{i,t}) < 0$ implies a negative effect. In addition, we consider the curve with $f_{\hat{\theta}}(\overline{v}_{i,t}) = 0$ and computed its 95% point-wise confidence intervals based on the variance estimates of $\hat{\theta}$. We plot this curve and its 95% point-wise confidence intervals in Fig. 4c and Fig. 4d for weekends and weekdays, respectively.

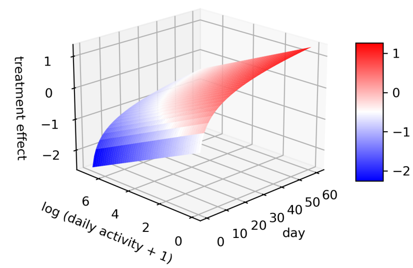
6. Concluding remarks

The multiplicative structural nested mean model describes the treatment effect on the mean shift of potential outcomes, where the shift can result from (i) the change of the probability of having positive outcomes or/and (ii) the change of the conditional mean given a positive outcome. However, the proposed model cannot distinguish and quantify these two pathways of treatment effects. In the point treatment setting, two-part models are used to model the treatment effect on (i) and (ii) separately; however, controversy exists regarding the causal interpretation of (ii) since it involves conditioning on a post-treatment variable, i.e., the potential outcome being positive. Although the proposed model only provides a coarsen description of the treatment effect, it avoids the controversy associated with the two-part model.

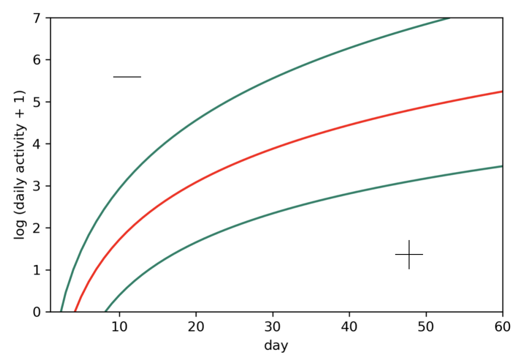
It is important to note that the proposed structural nested mean models are designed to address the causal effects of a series of treatments. We must add a caveat that our identification strategy assumes that all time-varying confounders are measured, which however is not verifiable based on the observed data but rely on subject matter experts to assess their plausibility. If some confounders have not been captured in our data, it can distort the causal interpretation. In these cases, one can conduct sensitivity analyses (Yang & Lok, 2018) to assess the impact of possible uncontrolled confounding.



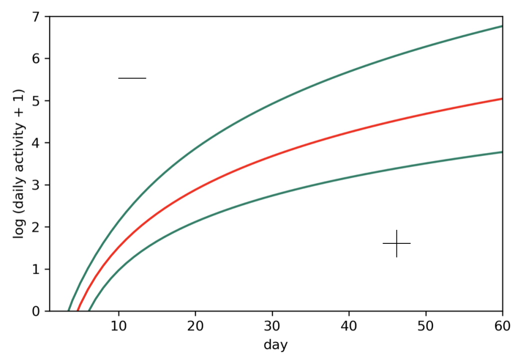
(a) The treatment effects $f_{\hat{\theta}}(\overline{v}_{i,t})$ (z-axis) on weekends.



(b) The treatment effects $f_{\hat{\theta}}(\overline{v}_{i,t})$ (z-axis) on weekdays.



(c) The curve for $f_{\hat{\theta}}(\overline{v}_{i,t}) = 0$ (middle line) and its 95% point-wise confidence intervals (outside lines) for weekends. The "+" / "-" indicates the area where the treatment effect is significantly positive/negative with a significance level of 0.05.



(d) The curve for $f_{\hat{\theta}}(\overline{v}_{i,t}) = 0$ (middle line) and its 95% point-wise confidence intervals (outside lines) for weekdays. The "+" / "-" indicates the area where the treatment effect is significantly positive/negative with a significance level of 0.05.

Fig. 4: Results of the multiplicative structural nested mean model in Freemium Mobile Game Data

In our current work, the ratio of conditional means of potential outcomes at time t is assumed to depend only on the current treatment a_t. We can also consider an elaborate model that accommodates not only the current but also previous treatments in the following form

$$\exp\{f^{(t)}_{\theta_0}(\overline{v}_{t-0})a_t + \cdots + f^{(t-K)}_{\theta_K}(\overline{v}_{t-K})a_{t-K}\},$$

where $f^{(t-k)}_{\theta_k}(\cdot)$ $(k = 0, \ldots, K)$ are known functions and $\theta = (\theta_0^T, \ldots, \theta_K^T)^T$ are the parameters of interest. Our framework requires the multiplicative structural nested mean model to be correctly specified, thus model assessment is critically important. The key insight is that we have a larger number of estimating functions than the number of parameters by varying c in equation (3), leading to the over-identification of model parameters. A goodness-of-fit test can be developed based on over-identification restrictions for model diagnosis (Yang & Lok, 2016). Another future direction is to extend the discrete-time setting in this work to continuous-time scenarios (Lok, 2008; Yang, 2021) that allow irregularly-spaced observations.

In practice, each promotion may not be identical. To extend our model to multiple categories of treatment, we define the treatment at time t as a dummy variable $a_t = (a_{t,1}, \ldots, a_{t,K})^T$, where $a_t = 0$ means no treatment. Our model can be adjusted in the following form

$$\exp\{f^{(1)}_{\theta_1}(\overline{v}_t)a_{t,1} + \cdots + f^{(K)}_{\theta_K}(\overline{v}_t)a_{t,K}\},$$

where $f_{\theta_k}^{(k)}(\cdot)$ $(k = 1, \ldots, K)$ are known functions and $\theta = (\theta_1^T, \ldots, \theta_K^T)^T$ are the parameters of interest. However, it requires a long enough follow-up time to collect sufficient data and get reliable estimation results.

Supplementary Material

Supplementary material available at *Biometrika* online includes theorem proofs and additional simulation results. The R codes for conducting the simulations reported in the paper can be downloaded from `https://www4.stat.ncsu.edu/~lu/programcodes.html`.

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[*Received on* 2 *January* 2017. *Editorial decision on* 8 *June* 2021]

Biometrika (2020), **103**, 1, *pp*. 1–14
Printed in Great Britain Advance Access publication on 31 July 2018

Supplementary Material for "Multiplicative Structural Nested Mean Model for Zero-Inflated Outcomes"

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The Supplementary Material contains two sections. § A provides the proof of the theories in the paper and § B gives additional simulation results.

A. PROOFS

A.1. *Proof of Theorem 1*

Before proving Theorem 1, we introduce the following two lemmas. The proofs of them are given at the end of the Supplementary Material.

LEMMA A1. *Assume Conditions 2 and 3 hold. Then, as $nT \to \infty$, we have*

$$\sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}} \|\mathbb{P}_n\psi(\theta,\eta) - \mathbb{P}\psi(\theta,\eta)\|_2 \xrightarrow{P} 0$$

and

$$\sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}} \|\mathbb{P}_n\dot{\psi}_\theta(\theta,\eta) - \mathbb{P}\dot{\psi}_\theta(\theta,\eta)\|_2 \xrightarrow{P} 0,$$

where $\dot{\psi}_\theta(\theta,\eta) = \partial\psi(\theta,\eta)/\partial\theta^T$.

LEMMA A2. *Assume Conditions 2–3 and 5–6 hold. Then, as $nT \to \infty$, we have*

$$\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0,\eta) \rightsquigarrow Z \text{ as random elements in } l^\infty(\mathcal{G}_{\eta_0}),$$

where "$\rightsquigarrow$" represents the weak convergence of a stochastic process, and $l^\infty(\mathcal{G}_{\eta_0})$ is the collection of all bounded functions $f : \mathcal{G}_{\eta_0} \to \mathbb{R}^p$. The limiting process $Z = \{Z(\eta) : \eta \in \mathcal{G}_{\eta_0}\}$ is a mean zero multivariate Gaussian process and the sample paths of Z belong to the set $UC(\mathcal{G}_{\eta_0}, \|\cdot\|_{2,P})$={$z \in l^\infty(\mathcal{G}_{\eta_0}) : z$ is uniformly continuous with respect to $\|\cdot\|_{2,P}$}.

First, we prove the consistency of $\hat{\theta}$ by showing $\|\mathbb{P}\psi(\hat{\theta},\eta_0)\|_2 \xrightarrow{P} 0$. Since we have

$$\begin{aligned}\|\mathbb{P}\psi(\hat{\theta},\eta_0)\|_2 &\le \|\mathbb{P}\psi(\hat{\theta},\eta_0) - \mathbb{P}\psi(\hat{\theta},\hat{\eta})\|_2 + \|\mathbb{P}\psi(\hat{\theta},\hat{\eta})\|_2 \\ &= \|\mathbb{P}\psi(\hat{\theta},\eta_0) - \mathbb{P}\psi(\hat{\theta},\hat{\eta})\|_2 + \|\mathbb{P}\psi(\hat{\theta},\hat{\eta}) - \mathbb{P}_n\psi(\hat{\theta},\hat{\eta})\|_2 \\ &\le \|\mathbb{P}\psi(\hat{\theta},\eta_0) - \mathbb{P}\psi(\hat{\theta},\hat{\eta})\|_2 + \sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}} \|\mathbb{P}_n\psi(\theta,\eta) - \mathbb{P}\psi(\theta,\eta)\|_2, \qquad \text{(A1)}\end{aligned}$$

it only needs to show that both terms in (A1) are negligible. By Taylor's expansion,

$$\begin{aligned}\|\psi_{0,t}(\hat\theta,\hat\eta)-\psi_{0,t}(\hat\theta,\eta_0)\|_2 &= \Big\|\frac{\partial\psi_{0,t}(\theta,\eta)}{\partial\eta^T}|_{\eta=\tilde\eta,\theta=\hat\theta}(\hat\eta-\eta_0)\Big\|_2\\ &\le \Big\|\frac{\partial\psi_{0,t}(\theta,\eta)}{\partial\eta^T}|_{\eta=\tilde\eta,\theta=\hat\theta}\Big\|_2\|\hat\eta-\eta_0\|_2,\end{aligned}$$

where $\|\tilde\eta-\eta_0\|_2<\|\hat\eta-\eta_0\|_2$. It follows that by the Cauchy-Schwartz inequality

$$\begin{aligned}&\|\mathbb{P}\psi(\hat\theta,\hat\eta)-\mathbb{P}\psi(\hat\theta,\eta_0)\|_2\\ &\le \mathbb{P}\|\psi(\hat\theta,\hat\eta)-\psi(\hat\theta,\eta_0)\|_2\\ &\le \mathbb{P}\left\{\Big\|\frac{\partial\psi(\theta,\eta)}{\partial\eta^T}|_{\eta=\tilde\eta,\theta=\hat\theta}\Big\|_2\|\hat\eta-\eta_0\|_2\right\}\\ &\le \frac{1}{T}\sum_{t=1}^{T}\left(\left[\mathbb{E}\left\{\Big\|\frac{\partial\psi_{0,t}(\theta,\eta)}{\partial\eta^T}|_{\eta=\tilde\eta,\theta=\hat\theta}\Big\|_2^2\right\}\right]^{1/2}\left[\mathbb{E}\left\{\|\hat\eta(\overline{V}_{0,t})-\eta_0(\overline{V}_{0,t})\|_2^2\right\}\right]^{1/2}\right)\\ &\le \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left\{\Big\|\frac{\partial\psi_{0,t}(\theta,\eta)}{\partial\eta^T}|_{\eta=\tilde\eta,\theta=\hat\theta}\Big\|_2^2\right\}\right]^{1/2}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left\{\|\hat\eta(\overline{V}_{0,t})-\eta_0(\overline{V}_{0,t})\|_2^2\right\}\right]^{1/2}\\ &\le b^*\|\hat\eta-\eta_0\|_{2,P}=o_P(1).\end{aligned} \qquad \text{(A2)}$$

By Lemma A1, we have

$$\sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}}\|\mathbb{P}_n\psi(\theta,\eta)-\mathbb{P}\psi(\theta,\eta)\|_2\xrightarrow{P}0 \text{ as } nT\to\infty.$$

Combining with (A1) and (A2), it yields that $\|\mathbb{P}\psi(\hat\theta,\eta_0)\|_2=o_P(1)$. Then, by the identification condition (Condition 1), we have $\hat\theta\xrightarrow{P}\theta_0$ as $nT\to\infty$.

Next, we prove the asymptotic distribution of $\hat\theta$. By Taylor expansion, we have

$$0=\sqrt{nT}\mathbb{P}_n\psi(\hat\theta,\hat\eta)=\sqrt{nT}\mathbb{P}_n\psi(\theta_0,\hat\eta)+\mathbb{P}_n\dot\psi_\theta(\tilde\theta,\hat\eta)\sqrt{nT}(\hat\theta-\theta_0),$$

where $\|\tilde\theta-\theta_0\|_2<\|\hat\theta-\theta_0\|_2$. By Lemma A1, we have

$$\sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}}\|\mathbb{P}_n\dot\psi_\theta(\theta,\eta)-\mathbb{P}\dot\psi_\theta(\theta,\eta)\|_2\xrightarrow{P}0 \text{ as } nT\to\infty.$$

It follows that

$$\mathbb{P}_n\dot\psi_\theta(\tilde\theta,\hat\eta)\xrightarrow{P}\mathbb{P}\dot\psi_\theta(\theta_0,\eta_0) \text{ as } nT\to\infty,$$

because $\tilde\theta\xrightarrow{P}\theta_0$ and $\hat\eta$ is also consistent by Condition 4. Therefore,

$$\begin{aligned}\sqrt{nT}(\hat\theta-\theta_0)&=-\left\{\mathbb{P}\dot\psi_\theta(\theta_0,\eta_0)\right\}^{-1}\sqrt{nT}\mathbb{P}_n\psi(\theta_0,\hat\eta)+o_P(1)\\ &=-\left\{\mathbb{P}\dot\psi_\theta(\theta_0,\eta_0)\right\}^{-1}\sqrt{nT}\left\{(\mathbb{P}_n-\mathbb{P})\psi(\theta_0,\hat\eta)+\mathbb{P}\psi(\theta_0,\hat\eta)\right\}+o_P(1).\end{aligned} \qquad \text{(A3)}$$

Moreover, we have

$$\begin{aligned}&\mathbb{P}\psi(\theta_0,\hat\eta)\\ &=\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\Big(c(\overline{V}_{0,t})\left[Y_{0,t}\exp\left\{-f_{\theta_0}(\overline{V}_{0,t})A_{0,t}\right\}-\hat\mu_1(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat\pi(\overline{V}_{0,t})\right.\\ &\left.-\hat\mu_0(\overline{V}_{0,t})\{1-\hat\pi(\overline{V}_{0,t})\}\right]\times\left\{I(A_{0,t}=1)-\hat\pi(\overline{V}_{0,t})\right\}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(c(\overline{V}_{0,t})\left[\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\pi_0(\overline{V}_{0,t}) + \mu_{0,0}(\overline{V}_{0,t})\{1-\pi_0(\overline{V}_{0,t})\}\right.\right.\\
&\quad \left.\left.-\hat{\mu}_1(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t}) - \hat{\mu}_0(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}\right]\times\left\{\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})\right\}\right)\\
&= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(c(\overline{V}_{0,t})\left[\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\pi_0(\overline{V}_{0,t}) - \mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t})\right.\right.\\
&\quad +\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t}) - \hat{\mu}_1(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t})\\
&\quad +\mu_{0,0}(\overline{V}_{0,t})\{1-\pi_0(\overline{V}_{0,t})\} - \mu_{0,0}(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}\\
&\quad \left.\left.+\mu_{0,0}(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\} - \hat{\mu}_0(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}\right]\times\left\{\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})\right\}\right).
\end{aligned}
$$

Applying the Cauchy-Schwartz inequality, after some calculations we have

$$
\begin{aligned}
&\|\mathbb{P}\psi(\theta_0,\hat{\eta})\|_2\\
&\leq \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(\left\|c(\overline{V}_{0,t})\left[\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\pi_0(\overline{V}_{0,t}) - \mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t})\right.\right.\right.\\
&\quad +\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t}) - \hat{\mu}_1(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t})\\
&\quad +\mu_{0,0}(\overline{V}_{0,t})\{1-\pi_0(\overline{V}_{0,t})\} - \mu_{0,0}(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}\\
&\quad \left.\left.\left.+\mu_{0,0}(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\} - \hat{\mu}_0(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}\right]\times\left\{\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})\right\}\right\|_2\right)\\
&\leq \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\|c(\overline{V}_{0,t})\|_2|\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\pi_0(\overline{V}_{0,t}) - \mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t})\right.\\
&\quad +\mu_{1,0}(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t}) - \hat{\mu}_1(\overline{V}_{0,t})\exp\{-f_{\theta_0}(\overline{V}_{0,t})\}\hat{\pi}(\overline{V}_{0,t})\\
&\quad +\mu_{0,0}(\overline{V}_{0,t})\{1-\pi_0(\overline{V}_{0,t})\} - \mu_{0,0}(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}\\
&\quad \left.+\mu_{0,0}(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\} - \hat{\mu}_0(\overline{V}_{0,t})\{1-\hat{\pi}(\overline{V}_{0,t})\}|\times|\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})|\right]\\
&\preceq \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\left\{|\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})| + |\mu_{1,0}(\overline{V}_{0,t})-\hat{\mu}_1(\overline{V}_{0,t})| + |\mu_{0,0}(\overline{V}_{0,t})-\hat{\mu}_0(\overline{V}_{0,t})|\right\}\right.\\
&\quad \left.\times|\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})|\right]\\
&\preceq \frac{1}{T}\sum_{t=1}^{T}\left\{\left(\mathbb{E}\left[\left\{|\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})| + |\mu_{1,0}(\overline{V}_{0,t})-\hat{\mu}_1(\overline{V}_{0,t})| + |\mu_{0,0}(\overline{V}_{0,t})-\hat{\mu}_0(\overline{V}_{0,t})|\right\}^2\right]\right)^{1/2}\right.\\
&\quad \left.\times\left(\mathbb{E}\left[\left\{\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})\right\}^2\right]\right)^{1/2}\right\}\\
&\preceq \left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\left\{|\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})| + |\mu_{1,0}(\overline{V}_{0,t})-\hat{\mu}_1(\overline{V}_{0,t})| + |\mu_{0,0}(\overline{V}_{0,t})-\hat{\mu}_0(\overline{V}_{0,t})|\right\}^2\right]\right)^{1/2}\\
&\quad \times\left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\left\{\pi_0(\overline{V}_{0,t})-\hat{\pi}(\overline{V}_{0,t})\right\}^2\right]\right)^{1/2}\\
&= \||\hat{\pi}-\pi_0| + |\hat{\mu}_1-\mu_{1,0}| + |\hat{\mu}_0-\mu_{0,0}|\|_{2,P}\times\|\hat{\pi}-\pi_0\|_{2,P}\\
&\leq \left(\|\hat{\pi}-\pi_0\|_{2,P} + \|\hat{\mu}_1-\mu_{1,0}\|_{2,P} + \|\hat{\mu}_0-\mu_{0,0}\|_{2,P}\right)\|\hat{\pi}-\pi_0\|_{2,P} = o_P\{(nT)^{-1/2}\},
\end{aligned}
$$

where we use the notation $\preceq$ to represent the left-hand side is bounded by a constant times the right-hand side. The last equality is due to Condition 4. This shows

$$
\sqrt{nT}\mathbb{P}\psi(\theta_0,\hat{\eta}) = o_p(1). \tag{A4}
$$

In addition, by Lemma A2, we have

$$\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \eta) \rightsquigarrow Z \text{ in } l^\infty(\mathcal{G}_{\eta_0}) \text{ as } nT \to \infty.$$

Since $\hat{\eta} \xrightarrow{P} \eta_0$ in the semimetric space $\mathcal{G}_{\eta_0}$ relative to the metric $\|\cdot\|_{2,P}$, it follows that

$$(\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \eta), \hat{\eta}) \rightsquigarrow (Z, \eta_0) \text{ in the space } l^\infty(\mathcal{G}_{\eta_0}) \times \mathcal{G}_{\eta_0} \text{ as } nT \to \infty.$$

Define a function $s : l^\infty(\mathcal{G}_{\eta_0}) \times \mathcal{G}_{\eta_0} \mapsto \mathbb{R}^p$ by $s(z, \eta) = z(\eta) - z(\eta_0)$. Notice that the function s is continuous at every point (z, η) such that $\eta \mapsto z(\eta)$ is continuous. By Lemma A2, almost all sample paths of Z are continuous on $\mathcal{G}_{\eta_0}$. Thus the function s is continuous at almost every point (Z, η_0). By the *continuous-mapping theorem*,

$$\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\left\{\psi(\theta_0, \hat{\eta}) - \psi(\theta_0, \eta_0)\right\} = s(\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \eta), \hat{\eta}) \rightsquigarrow s(Z, \eta_0) = 0.$$

Thus, we have

$$\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \hat{\eta}) = \sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \eta_0) + o_P(1).$$

Combining it with (A3) and (A4), we have

$$\sqrt{nT}(\hat{\theta} - \theta_0) = -\left\{\mathbb{P}\dot{\psi}_\theta(\theta_0, \eta_0)\right\}^{-1}\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \eta_0) + o_P(1).$$

Lastly, we only need to show

$$\sqrt{nT}(\mathbb{P}_n - \mathbb{P})\psi(\theta_0, \eta_0) \xrightarrow{d} MVN\left(0, \Sigma\right) \text{ as } nT \to \infty. \tag{A5}$$

For any integer $1 \leq g \leq nT$, let $i(g)$ be the quotient of $g + T$ divided by T, and $t(g)$ be the integer that satisfies

$$g = \{i(g) - 1\}T + t(g) \text{ and } 1 \leq t(g) \leq T.$$

Then proving (A5) is equivalent to proving

$$\sum_{g=1}^{nT}(nT)^{-1/2}\Sigma^{-1/2}\psi_{i(g),t(g)}(\theta_0, \eta_0) \xrightarrow{d} MVN(0, I) \text{ as } nT \to \infty. \tag{A6}$$

Let $\mathcal{F}_0 = \{L_{1,1}\}$ and iteratively define $\{\mathcal{F}_g\}_{1 \leq g \leq nT}$ as follows:

$$\mathcal{F}_g = \mathcal{F}_{g-1} \cup \left\{A_{i(g),t(g)}, Y_{i(g),t(g)}, L_{i(g),t(g)+1}\right\}, \text{ if } \quad t(g) < T$$
$$\mathcal{F}_g = \mathcal{F}_{g-1} \cup \left\{A_{i(g),T}, Y_{i(g),T}, L_{i(g)+1,1}\right\}, \text{ otherwise.}$$

By Proposition 2, we have $\mathbb{E}\left\{\psi_{i(g),t(g)}(\theta_0, \eta_0) \mid \mathcal{F}_{g-1}\right\} = 0$. Hence, the left-hand side of (A6) forms a martingale difference sequence with respect to the filtration $\{\sigma(\mathcal{F}_g)\}_{g \geq 1}$, where $\sigma(\mathcal{F}_g)$ stands for the σ-algebra generated by $\mathcal{F}_g$. To show the asymptotic normality, we apply the martingale central limit theorem for triangular arrays (Theorem 2.3 of McLeish et al. (1974)), which requires to verify the following two conditions:

Condition A1. $\max_{1 \leq g \leq nT} \|(nT)^{-1/2}\Sigma^{-1/2}\psi_{i(g),t(g)}(\theta_0, \eta_0)\|_2 \xrightarrow{P} 0$ as $nT \to \infty$.

Condition A2. $\frac{1}{nT}\sum_{g=1}^{nT}\Sigma^{-1/2}\psi_{i(g),t(g)}(\theta_0, \eta_0)\psi^T_{i(g),t(g)}(\theta_0, \eta_0)(\Sigma^{-1/2})^T \xrightarrow{P} I$ as $nT \to \infty$.

First, we have

$$\max_{1 \leq g \leq nT} \|(nT)^{-1/2}\Sigma^{-1/2}\psi_{i(g),t(g)}(\theta_0, \eta_0)\|_2$$
$$\leq (nT)^{-1/2}\|\Sigma^{-1/2}\|_2 \max_{1 \leq g \leq nT} \|\psi_{i(g),t(g)}(\theta_0, \eta_0)\|_2$$
$$\leq (nT)^{-1/2}\|\Sigma^{-1/2}\|_2 \max_{1 \leq g \leq nT} \left\{\|c(\overline{V}_{i(g),t(g)})\|_2 |I(A_{i(g),t(g)} = 1) - \pi_0(\overline{V}_{i(g),t(g)})|\right.$$

$$
\begin{aligned}
&\times|Y_{i(g),t(g)}\exp\{-f_{\theta_0}(\overline{V}_{i(g),t(g)})A_{i(g),t(g)}\} - \mu_{1,0}(\overline{V}_{i(g),t(g)})\exp\{-f_{\theta_0}(\overline{V}_{i(g),t(g)})\}\pi_0(\overline{V}_{i(g),t(g)})\\
&-\mu_{0,0}(\overline{V}_{i(g),t(g)})\{1-\pi_0(\overline{V}_{i(g),t(g)})\}|\}\\
&= O_p\{(nT)^{-1/2}\},
\end{aligned}
$$

which completes the proof of Condition A1.

To verify Condition A2, we have

$$
\begin{aligned}
&\Big\|(nT)^{-1}\sum_{g=1}^{nT}\Sigma^{-1/2}\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0)(\Sigma^{-1/2})^T - I\Big\|_2\\
&= \Big\|\Sigma^{-1/2}\left\{(nT)^{-1}\sum_{g=1}^{nT}\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) - \Sigma\right\}(\Sigma^{-1/2})^T\Big\|_2\\
&\le \|\Sigma^{-1/2}\|_2^2\Big\|(nT)^{-1}\sum_{g=1}^{nT}\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) - \Sigma\Big\|_2.
\end{aligned}
$$

It suffices to show

$$
\Big\|\frac{1}{nT}\sum_{g=1}^{nT}\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) - \Sigma\Big\|_2 = o_P(1).
$$

Define $M_g = \psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) - \mathbb{E}\left\{\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) \mid \mathcal{F}_{g-1}\right\}$. Then $\{M_g\}_{g\ge 1}$ forms a martingale difference sequence with respect to the filtration $\{\sigma(\mathcal{F}_g)\}_{g\ge 1}$. Since $\mathbb{E}\left(M_g M^T_{g'}\right) = 0$ for $g \ne g'$ and $\mathbb{E}\left(M_g M_g^T\right)$ is bounded for all g, we have $\|(nT)^{-1}\sum_{g=1}^{nT} M_g\|_2 \xrightarrow{P} 0$ as $nT \to \infty$. That is,

$$
\Big\|\frac{1}{nT}\sum_{g=1}^{nT}\left[\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) - \mathbb{E}\left\{\psi_{i(g),t(g)}(\theta_0,\eta_0)\psi^T_{i(g),t(g)}(\theta_0,\eta_0) \mid \mathcal{F}_{g-1}\right\}\right]\Big\|_2 \xrightarrow{P} 0.
$$

This proves Condition A2. The proof of Theorem 1 is hence completed.

A.2. *Proof of Theorem 2*

In the case of timewise randomization, we can establish similar results as for Lemmas A1 and A2 under modified conditions. The remaining proof is similar to that of Theorem 1, which is omitted here.

A.3. *Proof of Lemma A1*

For any integer $1 \le g \le nT$, let $i(g)$ be the quotient of $g + T$ divided by T, and $t(g)$ be the integer that satisfies

$$
g = (i(g)-1)T + t(g) \text{ and } 1 \le t(g) \le T.
$$

Let $\mathcal{F}_0 = \{L_{1,1}\}$, and $\{\mathcal{F}_g\}_{1\le g\le nT}$ iteratively defined as follows:

$$
\begin{aligned}
\mathcal{F}_g &= \mathcal{F}_{g-1} \cup \left\{A_{i(g),t(g)}, Y_{i(g),t(g)}, L_{i(g),t(g)+1}\right\}, \text{ if } \quad t(g) < T\\
\mathcal{F}_g &= \mathcal{F}_{g-1} \cup \left\{A_{i(g),T}, Y_{i(g),T}, L_{i(g)+1,1}\right\}, \text{ otherwise.}
\end{aligned}
$$

For any $(\theta,\eta) \in \mathcal{U}$, define

$$
m_g(\theta,\eta) = \psi_{i(g),t(g)}(\theta,\eta) - \mathbb{E}\{\psi_{i(g),t(g)}(\theta,\eta) \mid \mathcal{F}_{g-1}\}.
$$

Then for any fixed (θ,η), $m_g(\theta,\eta)$ is a martingale difference sequence adapted to $\{\sigma(\mathcal{F}_g)\}_{g\ge 1}$. By the continuousness of $m_g(\cdot)$ in θ and η and Condition 2, we have that for any $\delta > 0$, there exists a finite ϵ-net

$\mathcal{U}_\epsilon$ such that

$$\sup_{(\theta,\eta)\in\mathcal{U}} \frac{1}{nT}\Big\|\sum_{g=1}^{nT} m_g(\theta,\eta)\Big\|_2 \leq \sup_{(\theta,\eta)\in\mathcal{U}_\epsilon} \frac{1}{nT}\Big\|\sum_{g=1}^{nT} m_g(\theta,\eta)\Big\|_2 + \delta. \quad \text{(A7)}$$

For any $(\theta,\eta)\in\mathcal{U}_\epsilon$, define

$$W_{col,nT} = \sum_{g=1}^{nT} \mathbb{E}\left\{m_g(\theta,\eta)m_g^T(\theta,\eta) \mid \mathcal{F}_{g-1}\right\} \text{ and}$$

$$W_{row,nT} = \sum_{g=1}^{nT} \mathbb{E}\left\{m_g^T(\theta,\eta)m_g(\theta,\eta) \mid \mathcal{F}_{g-1}\right\}.$$

By Condition 3, we can show that for any $(\theta,\eta)\in\mathcal{U}$

$$\max\left\{\|W_{col,nT}\|_2, \|W_{row,nT}\|_2\right\} \leq nT\sigma^2 \quad a.s.$$
$$\|m_g(\theta,\eta)\|_2 \leq r \quad a.s.$$

for some constant σ^2 and r. Then by the *martingale concentration inequality* (Tropp et al., 2011), we have

$$\mathbb{P}\left\{\Big\|\sum_{g=1}^{nT} m_g(\theta,\eta)\Big\|_2 \geq \tau\right\} \leq (1+p)\exp\left(\frac{-\tau^2}{\sigma^2 nT + r\tau/3}\right).$$

Setting $\tau = \sqrt{nT\log nT}$, we can show that the following event occurs with a probability of at least $1 - O(1/nT)$ for any $(\theta,\eta)\in\mathcal{U}$,

$$\Big\|\sum_{g=1}^{nT} m_g(\theta,\eta)\Big\|_2 \preceq \sqrt{nT\log nT}.$$

Since $\mathcal{U}_\epsilon$ is finite, we have

$$\sup_{(\theta,\eta)\in\mathcal{U}_\epsilon} \Big\|\sum_{g=1}^{nT} m_g(\theta,\eta)\Big\|_2 \preceq \sqrt{nT\log nT},$$

and hence

$$\sup_{(\theta,\eta)\in\mathcal{U}_\epsilon} \Big\|\frac{1}{nT}\sum_{g=1}^{nT} m_g(\theta,\eta)\Big\|_2 = O_P\Big(\sqrt{\frac{\log nT}{nT}}\Big) = o_P(1).$$

By (A7), we obtain

$$\sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}} \|\mathbb{P}_n\psi(\theta,\eta) - \mathbb{P}\psi(\theta,\eta)\|_2 = o_P(1).$$

In a similar way, we can show

$$\sup_{\theta\in\Theta,\eta\in\mathcal{G}_{\eta_0}} \|\mathbb{P}_n\dot{\psi}_\theta(\theta,\eta) - \mathbb{P}\dot{\psi}_\theta(\theta,\eta)\|_2 = o_P(1).$$

A.4. *Proof of Lemma A2*

We prove this lemma by applying the uniform central limit theorem for function-indexed martingale difference arrays (Theorem 2 of Bae et al. (2010)). Specifically, proving Lemma A2 is equivalent to proving

$$\frac{1}{\sqrt{nT}}\sum_{g=1}^{nT} m_g(\eta) \rightsquigarrow Z \text{ in } l^\infty(\mathcal{G}_{\eta_0}) \text{ as } nT\to\infty.$$

Note that for any fixed η, $m_g(\eta)$ is a martingale difference sequence adapted to $\{\sigma(\mathcal{F}_g)\}_{g\geq 1}$. Let $m_g^{(j)}(\eta)$ denote the jth component of $m_g(\eta)$. To show the weak convergence, we need to verify the following conditions:

Condition A3. $\mathcal{G}_{\eta_0}$ has uniformly integrable entropy.

Condition A4. For $j = 1, \ldots, p$, there exists a constant M such that as $nT \to \infty$

$$P\left(\sup_{\eta,\eta' \in \mathcal{G}_{\eta_0}} \frac{(nT)^{-1}\sum_{g=1}^{nT} \mathbb{E}\left[\left\{m_g^{(j)}(\eta) - m_g^{(j)}(\eta')\right\}^2 \mid \overline{V}_g\right]}{\|\eta - \eta'\|_{2,P}^2} \geq M\right) \to 0.$$

Condition A5. For $j = 1, \ldots, p$, $(nT)^{-1}\sum_{g=1}^{nT} \mathbb{E}\left[\{m_g^{(j)}(F)\}^2 1\left\{(nT)^{-1/2} m_g^{(j)}(F) > \epsilon\right\}\right] \xrightarrow{P} 0$ as $nT \to \infty$, where F is an envelope function for the class of functions $\{\eta : \eta \in \mathcal{G}_{\eta_0}\}$.

Condition A6. $(nT)^{-1}\sum_{g=1}^{nT} \mathbb{E}\left\{m_g(\eta) m_g^T(\eta) \mid \mathcal{F}_{g-1}\right\} \xrightarrow{P} \Sigma(\eta)$ as $nT \to \infty$ for each $\eta \in \mathcal{G}_{\eta_0}$, where $\Sigma(\eta)$ is a positive definite matrix.

Here, Conditions A3, A4, and A6 are provided by Conditions 2, 6, and 5, respectively, and their validity is discussed after introducing these conditions. So we only need to show Condition A5. An envelop function F can be chosen as a constant vector $(b_0, b_0, b_0)^T$ due to Condition 3(iii). Moreover, we can show that $\|m_g(F)\|_2$ is bounded almost surely. Therefore, for $j = 1, \ldots, p$,

$$\mathbb{E}\left[\{m_g^{(j)}(F)\}^2 1\left\{(nT)^{-1/2} m_g^{(j)}(F) > \epsilon\right\}\right] \preceq P\left\{(nT)^{-1/2} m_g^{(j)}(F) > \epsilon\right\} \xrightarrow{P} 0,$$

which completes the proof of Condition A5.

B. Additional Simulations

B.1. *Comparison of Condition Outcome Means Estimation Methods*

In the proposed method, the conditional outcome means $\mu_{0,0}, \mu_{1,0}$ are nonparametrically estimated in two parts using generalized additive models. We want to see how the proposed method compares with the approach where the conditional outcome means are nonparametrically estimated in one part and the approach where the conditional outcome means are modeled parametrically in two parts with incorrect function forms. We consider the setting of individual randomization and the covariates-dependent propensity score $\pi_0(\overline{v}_{i,t}) = \{1 + \exp(0.5 - 0.5 l_{i,t}^{(2)} - 0.8 l_{i,t}^{(3)})\}^{-1}$. We generate the potential outcomes $Y_{i,t}^{(\overline{a}_{i,T})}$ from a zero-inflated log-normal distribution

$$(1 - p_{i,t}^{(\overline{a}_{i,T})}) I(Y_{i,t}^{(\overline{a}_{i,T})} = 0) + p_{i,t}^{(\overline{a}_{i,T})} \text{ log-normal } (\nu_{i,t}^{(\overline{a}_{i,T})}, 0.5^2),$$

where $p_{i,t}^{(\overline{a}_{i,T})} = \exp(\beta_p^T \tilde{l}_{i,t} + \gamma_p^T \tilde{l}_{i,t} a_{i,t})$ and $\nu_{i,t}^{(\overline{a}_{i,T})} = \beta_\nu^T \tilde{l}_{i,t} + \gamma_\nu^T \tilde{l}_{i,t} a_{i,t}$ with $\tilde{l}_{i,t} = (1, l_{i,t}^T)^T$. Then, $f_{\theta_0}(\overline{v}_{i,t}) = \theta_0^T \tilde{l}_{i,t}$ with $\theta_0 = \gamma_p + \gamma_\nu$. The parameters are chosen as below such that the zero proportion of the observed outcomes is around 90%.

$$\begin{aligned}
&\beta_p = (-0.6, -0.8, -0.5, -1)^T, \beta_\nu = (0, 0.3, 0.5, 0.5)^T, \\
&\gamma_p = (0.5, 0.5, 0, 0)^T, \gamma_\nu = (0.1, 0, 0.4, 0.5)^T, \\
&\theta_0 = (0.6, 0.5, 0.4, 0.5)^T.
\end{aligned}$$

The propensity score is nonparametrically fitted by a generalized additive model in all three methods. In the one-part nonparametric method, the conditional outcome means are estimated by a generalized additive model, while in the two-part parametric method the two parts of the conditional outcome means are respectively modeled by logistic regression and linear regression. The results in Table A1 show that the proposed estimators are more accurate and efficient compared with the others when the sample size is small. The advantage becomes small when the sample size gets large.

Table A1: Simulation results with nonparametric propensity score and $T = 60$ under individual randomization

Estimation method for μ_0, μ_1	n	Parameter	Opt. c				Sim. c			
			Mean	SD	SE	C.P.	Mean	SD	SE	C.P.
two-parts, nonparametric	1000	$\theta_0^{(0)} = 0.6$	0.599	0.055	0.054	95.2	0.601	0.087	0.088	94.8
		$\theta_0^{(1)} = 0.5$	0.501	0.024	0.024	95.7	0.501	0.035	0.036	95.3
		$\theta_0^{(2)} = 0.4$	0.406	0.066	0.065	94.4	0.404	0.087	0.081	92.8
		$\theta_0^{(3)} = 0.5$	0.502	0.114	0.113	94.8	0.504	0.135	0.126	93.9
	400	$\theta_0^{(0)} = 0.6$	0.600	0.086	0.088	96.3	0.599	0.123	0.116	93.9
		$\theta_0^{(1)} = 0.5$	0.500	0.041	0.042	96.0	0.499	0.054	0.052	94.6
		$\theta_0^{(2)} = 0.4$	0.410	0.106	0.103	93.9	0.417	0.133	0.125	92.1
		$\theta_0^{(3)} = 0.5$	0.507	0.172	0.172	96.3	0.523	0.243	0.223	93.0
	200	$\theta_0^{(0)} = 0.6$	0.594	0.124	0.123	95.1	0.607	0.215	0.205	93.9
		$\theta_0^{(1)} = 0.5$	0.504	0.065	0.066	95.8	0.498	0.074	0.075	95.1
		$\theta_0^{(2)} = 0.4$	0.404	0.154	0.149	93.6	0.433	0.197	0.166	89.5
		$\theta_0^{(3)} = 0.5$	0.533	0.265	0.263	94.8	0.531	0.290	0.271	93.8
one-part, nonparametric	1000	$\theta_0^{(0)} = 0.6$	0.601	0.054	0.055	95.9	0.599	0.113	0.114	95.1
		$\theta_0^{(1)} = 0.5$	0.501	0.026	0.027	96.5	0.501	0.037	0.037	95.7
		$\theta_0^{(2)} = 0.4$	0.399	0.063	0.063	95.0	0.402	0.088	0.082	92.5
		$\theta_0^{(3)} = 0.5$	0.501	0.100	0.103	94.4	0.509	0.122	0.123	95.0
	400	$\theta_0^{(0)} = 0.6$	0.605	0.137	0.142	95.5	0.588	0.273	0.261	93.0
		$\theta_0^{(1)} = 0.5$	0.501	0.056	0.057	96.0	0.504	0.078	0.076	94.0
		$\theta_0^{(2)} = 0.4$	0.406	0.106	0.103	94.4	0.416	0.131	0.122	92.3
		$\theta_0^{(3)} = 0.5$	0.497	0.147	0.149	96.2	0.514	0.219	0.208	93.6
	200	$\theta_0^{(0)} = 0.6$	0.594	0.298	0.299	94.9	0.585	0.421	0.380	91.8
		$\theta_0^{(1)} = 0.5$	0.505	0.092	0.092	95.2	0.511	0.123	0.113	91.9
		$\theta_0^{(2)} = 0.4$	0.417	0.158	0.144	93.0	0.420	0.186	0.161	90.0
		$\theta_0^{(3)} = 0.5$	0.527	0.250	0.238	93.8	0.517	0.298	0.287	93.8
two-parts, misspecified parametric	1000	$\theta_0^{(0)} = 0.6$	0.600	0.108	0.110	95.2	0.598	0.156	0.156	96.5
		$\theta_0^{(1)} = 0.5$	0.501	0.037	0.037	94.9	0.501	0.050	0.050	95.2
		$\theta_0^{(2)} = 0.4$	0.402	0.068	0.069	94.9	0.406	0.088	0.083	94.1
		$\theta_0^{(3)} = 0.5$	0.503	0.111	0.108	94.4	0.507	0.133	0.128	94.1
	400	$\theta_0^{(0)} = 0.6$	0.578	0.212	0.211	96.2	0.579	0.242	0.241	95.2
		$\theta_0^{(1)} = 0.5$	0.507	0.067	0.067	96.6	0.506	0.075	0.074	94.5
		$\theta_0^{(2)} = 0.4$	0.411	0.112	0.111	95.4	0.421	0.138	0.124	90.1
		$\theta_0^{(3)} = 0.5$	0.521	0.172	0.176	95.5	0.514	0.197	0.201	94.3
	200	$\theta_0^{(0)} = 0.6$	0.561	0.332	0.318	96.3	0.580	0.374	0.358	95.9
		$\theta_0^{(1)} = 0.5$	0.515	0.120	0.113	96.3	0.509	0.107	0.105	96.0
		$\theta_0^{(2)} = 0.4$	0.432	0.171	0.177	96.6	0.424	0.198	0.172	90.7
		$\theta_0^{(3)} = 0.5$	0.519	0.241	0.245	95.7	0.524	0.311	0.286	93.6

B.2. *Simulations under Incorrect Treatment Effect Model*

Our framework requires that the treatment effect model is correctly specified. We want to test how the proposed method performs when the treatment effect model is misspecified. We consider two cases of misspecified treatment effect model. In both cases, the treatment is generated from a constant propensity score of 0.5.

In the first case, treatment $a_{i,t-1}$ is involved in the true treatment effect function as an additional covariate, i.e.

$$\frac{\mathbb{E}(Y_{i,t}^{(\overline{a}_{i,t})} \mid \overline{A}_{i,t} = \overline{a}_{i,t}, \overline{L}_{i,t} = \overline{l}_{i,t})}{\mathbb{E}(Y_{i,t}^{(\overline{0}_{i,t})} \mid \overline{A}_{i,t} = \overline{a}_{i,t}, \overline{L}_{i,t} = \overline{l}_{i,t})} = \exp\{f_\theta(\overline{v}_{i,t})a_{i,t}\} \quad t = 1, \ldots, T;\, i = 1, \ldots, n \tag{A1}$$

where

$$f_{\theta_0}(\overline{v}_{i,t}) = \theta_0^T \tilde{l}_{i,t} + 0.2a_{i,t-1}.$$

When the treatment effect model is correctly specified, our proposed estimator is still unbiased and the resulting variance estimation is consistent (Table A2). However, if we do not consider the term $a_{i,t-1}$ in constructing the treatment effect model, we will have a biased estimator for the intercept (Table A3).

In the second case, we generate the potential outcomes from a zero-inflated log-normal distribution

$$(1 - p_{i,t}^{(\overline{a}_{i,t})})I(Y_{i,t}^{(\overline{a}_{i,t})} = 0) + p_{i,t}^{(\overline{a}_{i,t})} \text{ log-normal } (\nu_{i,t}^{(\overline{a}_{i,t})}, 0.5^2),$$

where $p_{i,t}^{(\overline{a}_{i,t})} = \exp(\beta_p^T \tilde{l}_{i,t} + 0.4a_{i,t} + 0.2a_{i,t-1})$ and $\nu_{i,t}^{(\overline{a}_{i,t})} = \beta_\nu^T \tilde{l}_{i,t} + 0.3a_{i,t} + 0.1a_{i,t-1}$ with $\tilde{l}_{i,t} = (1, l_{i,t}^T)^T$, so the true treatment effect model should be

$$\frac{\mathbb{E}(Y_{i,t}^{(\overline{a}_{i,t})} \mid \overline{A}_{i,t} = \overline{a}_{i,t}, \overline{L}_{i,t} = \overline{l}_{i,t})}{\mathbb{E}(Y_{i,t}^{(\overline{0}_{i,t})} \mid \overline{A}_{i,t} = \overline{a}_{i,t}, \overline{L}_{i,t} = \overline{l}_{i,t})} = \exp\{0.7a_{i,t} + 0.3a_{i,t-1}\} \quad t = 1, \ldots, T;\, i = 1, \ldots, n$$

However, we consider an incorrect treatment effect function $f_\theta(\overline{v}_{i,t}) = \theta^{(0)}$. Interestingly, Table A4 shows that we have a nearly unbiased estimator of the treatment effect for $a_{i,t}$, but we add the caveat that this robustness may not extend to general settings and that an incorrect treatment effect model obfuscates the interpretation.

B.3. *Variance Estimation Simulation Results*

In the main paper, we showed point estimation results by boxplots. Here we provide the variance estimation results of the proposed estimator under the same settings. Tables A5–A8 correspond to the same setting as Fig.1–3 in § 4.

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[*Received on* 2 *January* 2017. *Editorial decision on* 1 *April* 2017]

Table A2: Simulation results with correct treatment effect function (case 1)

n	Parameter	Mean	SD	SE	C.P.
1000	$\theta_0^{(0)} = 0.6$	0.596	0.151	0.150	95
	$\theta_0^{(1)} = 0.5$	0.502	0.042	0.043	95
	$\theta_0^{(2)} = 0.4$	0.403	0.068	0.066	94
	$\theta_0^{(3)} = 0.5$	0.504	0.116	0.111	94
	$\theta_0^{(4)} = 0.5$	0.201	0.105	0.106	96
400	$\theta_0^{(0)} = 0.6$	0.605	0.113	0.113	95
	$\theta_0^{(1)} = 0.5$	0.499	0.051	0.052	95
	$\theta_0^{(2)} = 0.4$	0.410	0.107	0.108	96
	$\theta_0^{(3)} = 0.5$	0.501	0.181	0.182	95
	$\theta_0^{(4)} = 0.2$	0.198	0.142	0.142	95
200	$\theta_0^{(0)} = 0.6$	0.603	0.211	0.206	94
	$\theta_0^{(1)} = 0.5$	0.504	0.080	0.079	96
	$\theta_0^{(2)} = 0.4$	0.412	0.159	0.141	93
	$\theta_0^{(3)} = 0.5$	0.518	0.235	0.232	95
	$\theta_0^{(4)} = 0.2$	0.196	0.228	0.225	95

Table A3: Simulation results with incorrect treatment effect function (case 1)

n	Parameter	Mean	SD
1000	$\theta_0^{(0)} = 0.6$	0.666	0.170
	$\theta_0^{(1)} = 0.5$	0.509	0.048
	$\theta_0^{(2)} = 0.4$	0.418	0.066
	$\theta_0^{(3)} = 0.5$	0.500	0.119
400	$\theta_0^{(0)} = 0.6$	0.642	0.145
	$\theta_0^{(1)} = 0.5$	0.516	0.050
	$\theta_0^{(2)} = 0.4$	0.426	0.104
	$\theta_0^{(3)} = 0.5$	0.515	0.159
200	$\theta_0^{(0)} = 0.6$	0.690	0.281
	$\theta_0^{(1)} = 0.5$	0.509	0.089
	$\theta_0^{(2)} = 0.4$	0.423	0.158
	$\theta_0^{(3)} = 0.5$	0.499	0.243

Table A4: Simulation results with incorrect treatment effect function (case 2)

n	Parameter	Mean	SD
1000	$\theta_0^{(0)} = 0.7$	0.704	0.062
400	$\theta_0^{(0)} = 0.7$	0.702	0.097
200	$\theta_0^{(0)} = 0.7$	0.705	0.117

Table A5: Simulation results with same directional treatment effect and $T = 60$ under individual randomization

Propensity score	n	Parameter	Opt. c				Sim. c			
			Mean	SD	SE	C.P.	Mean	SD	SE	C.P.
constant	1000	$\theta_0^{(0)} = 0.30$	0.300	0.053	0.052	94	0.299	0.057	0.056	94
		$\theta_0^{(1)} = 0.15$	0.150	0.016	0.016	95	0.150	0.017	0.017	95
		$\theta_0^{(2)} = 0.05$	0.050	0.017	0.017	96	0.050	0.019	0.019	96
		$\theta_0^{(3)} = 0.08$	0.081	0.035	0.035	95	0.081	0.038	0.037	95
	400	$\theta_0^{(0)} = 0.30$	0.302	0.087	0.082	94	0.302	0.093	0.089	95
		$\theta_0^{(1)} = 0.15$	0.150	0.026	0.025	95	0.149	0.028	0.027	95
		$\theta_0^{(2)} = 0.05$	0.050	0.027	0.027	96	0.051	0.029	0.030	96
		$\theta_0^{(3)} = 0.08$	0.080	0.057	0.055	94	0.081	0.061	0.059	94
	200	$\theta_0^{(0)} = 0.30$	0.299	0.116	0.117	95	0.304	0.127	0.126	94
		$\theta_0^{(1)} = 0.15$	0.150	0.035	0.036	97	0.149	0.038	0.038	96
		$\theta_0^{(2)} = 0.05$	0.052	0.039	0.039	95	0.052	0.043	0.043	95
		$\theta_0^{(3)} = 0.08$	0.078	0.079	0.078	95	0.077	0.084	0.083	94
covariates dependent	1000	$\theta_0^{(0)} = 0.30$	0.302	0.053	0.056	96	0.302	0.063	0.063	94
		$\theta_0^{(1)} = 0.15$	0.150	0.016	0.017	97	0.150	0.018	0.018	96
		$\theta_0^{(2)} = 0.05$	0.050	0.020	0.019	95	0.050	0.022	0.022	95
		$\theta_0^{(3)} = 0.08$	0.079	0.038	0.037	95	0.080	0.044	0.043	96
	400	$\theta_0^{(0)} = 0.30$	0.297	0.081	0.078	95	0.297	0.093	0.096	96
		$\theta_0^{(1)} = 0.15$	0.151	0.027	0.026	95	0.150	0.030	0.029	95
		$\theta_0^{(2)} = 0.05$	0.052	0.036	0.036	95	0.051	0.034	0.034	94
		$\theta_0^{(3)} = 0.08$	0.078	0.061	0.062	95	0.083	0.066	0.067	96
	200	$\theta_0^{(0)} = 0.30$	0.295	0.136	0.130	94	0.301	0.144	0.140	94
		$\theta_0^{(1)} = 0.15$	0.151	0.039	0.038	95	0.150	0.043	0.042	94
		$\theta_0^{(2)} = 0.05$	0.050	0.047	0.045	94	0.052	0.048	0.048	95
		$\theta_0^{(3)} = 0.08$	0.088	0.085	0.085	95	0.079	0.093	0.093	96

Table A6: Simulation results with opposite directional treatment effect and $T = 60$ under individual randomization

Propensity score	n	Parameter	Opt. c Mean	Opt. c SD	Opt. c SE	Opt. c C.P.	Sim. c Mean	Sim. c SD	Sim. c SE	Sim. c C.P.
constant	1000	$\theta_0^{(0)} = -0.10$	-0.102	0.053	0.052	94	-0.100	0.060	0.059	94
		$\theta_0^{(1)} = -0.15$	-0.149	0.016	0.016	95	-0.150	0.018	0.018	95
		$\theta_0^{(2)} = 0.05$	0.051	0.018	0.017	95	0.050	0.020	0.019	94
		$\theta_0^{(3)} = 0.08$	0.080	0.035	0.035	95	0.079	0.038	0.038	95
	400	$\theta_0^{(0)} = -0.10$	-0.099	0.082	0.083	94	-0.099	0.094	0.094	95
		$\theta_0^{(1)} = -0.15$	-0.150	0.026	0.026	95	-0.150	0.028	0.028	95
		$\theta_0^{(2)} = 0.05$	0.050	0.027	0.028	95	0.049	0.031	0.030	95
		$\theta_0^{(3)} = 0.08$	0.082	0.053	0.056	95	0.082	0.057	0.060	96
	200	$\theta_0^{(0)} = -0.10$	-0.095	0.114	0.117	95	-0.090	0.130	0.133	95
		$\theta_0^{(1)} = -0.15$	-0.152	0.034	0.036	96	-0.153	0.038	0.040	96
		$\theta_0^{(2)} = 0.05$	0.051	0.038	0.039	95	0.050	0.043	0.043	94
		$\theta_0^{(3)} = 0.08$	0.081	0.080	0.079	94	0.080	0.086	0.084	95
covariates dependent	1000	$\theta_0^{(0)} = -0.10$	-0.104	0.051	0.052	95	-0.103	0.056	0.056	96
		$\theta_0^{(1)} = -0.15$	-0.149	0.016	0.016	95	-0.149	0.018	0.018	95
		$\theta_0^{(2)} = 0.05$	0.051	0.018	0.018	96	0.050	0.018	0.018	95
		$\theta_0^{(3)} = 0.08$	0.082	0.036	0.036	95	0.081	0.036	0.037	95
	400	$\theta_0^{(0)} = -0.10$	-0.105	0.101	0.099	94	-0.096	0.089	0.090	96
		$\theta_0^{(1)} = -0.15$	-0.150	0.028	0.026	94	-0.152	0.028	0.028	95
		$\theta_0^{(2)} = 0.05$	0.051	0.026	0.026	95	0.052	0.027	0.028	95
		$\theta_0^{(3)} = 0.08$	0.085	0.060	0.060	96	0.078	0.059	0.059	95
	200	$\theta_0^{(0)} = -0.10$	-0.099	0.115	0.114	95	-0.098	0.129	0.131	96
		$\theta_0^{(1)} = -0.15$	-0.150	0.039	0.038	94	-0.151	0.039	0.039	96
		$\theta_0^{(2)} = 0.05$	0.051	0.038	0.037	95	0.052	0.037	0.039	96
		$\theta_0^{(3)} = 0.08$	0.080	0.078	0.081	96	0.085	0.087	0.083	93

Table A7: Simulation results with same directional treatment effect and $n = 400$ under timewise randomization

T	Parameter	Opt. c				Sim. c			
		Mean	SD	SE	C.P.	Mean	SD	SE	C.P.
1000	$\theta_0^{(0)} = 0.30$	0.300	0.057	0.055	93	0.301	0.053	0.052	95
	$\theta_0^{(1)} = 0.15$	0.150	0.009	0.009	92	0.150	0.009	0.009	94
	$\theta_0^{(2)} = 0.05$	0.050	0.011	0.011	95	0.050	0.012	0.012	95
	$\theta_0^{(3)} = 0.08$	0.080	0.020	0.021	96	0.079	0.021	0.022	96
250	$\theta_0^{(0)} = 0.30$	0.298	0.089	0.075	86	0.300	0.080	0.072	90
	$\theta_0^{(1)} = 0.15$	0.151	0.018	0.016	88	0.150	0.017	0.015	91
	$\theta_0^{(2)} = 0.05$	0.050	0.020	0.020	94	0.050	0.021	0.020	94
	$\theta_0^{(3)} = 0.08$	0.080	0.035	0.035	95	0.079	0.037	0.036	94
60	$\theta_0^{(0)} = 0.30$	0.293	0.136	0.098	81	0.296	0.121	0.098	87
	$\theta_0^{(1)} = 0.15$	0.150	0.038	0.028	81	0.150	0.034	0.029	89
	$\theta_0^{(2)} = 0.05$	0.052	0.038	0.036	92	0.052	0.038	0.036	92
	$\theta_0^{(3)} = 0.08$	0.085	0.060	0.060	95	0.083	0.064	0.061	93

Table A8: Simulation results with opposite directional treatment effect and $n = 400$ under timewise randomization

T	Parameter	Opt. c				Sim. c			
		Mean	SD	SE	C.P.	Mean	SD	SE	C.P.
1000	$\theta_0^{(0)} = -0.10$	-0.100	0.053	0.052	95	-0.099	0.055	0.054	95
	$\theta_0^{(1)} = -0.15$	-0.150	0.009	0.009	95	-0.150	0.009	0.009	94
	$\theta_0^{(2)} = 0.05$	0.050	0.011	0.011	95	0.050	0.012	0.012	95
	$\theta_0^{(3)} = 0.08$	0.080	0.020	0.021	96	0.080	0.022	0.022	95
250	$\theta_0^{(0)} = -0.10$	-0.102	0.083	0.073	90	-0.100	0.080	0.075	94
	$\theta_0^{(1)} = -0.15$	-0.150	0.017	0.015	92	-0.150	0.017	0.016	93
	$\theta_0^{(2)} = 0.05$	0.050	0.021	0.020	94	0.050	0.021	0.020	94
	$\theta_0^{(3)} = 0.08$	0.080	0.035	0.035	95	0.079	0.037	0.037	94
60	$\theta_0^{(0)} = -0.10$	-0.097	0.140	0.100	81	-0.096	0.132	0.103	86
	$\theta_0^{(1)} = -0.15$	-0.151	0.040	0.029	84	-0.151	0.037	0.030	88
	$\theta_0^{(2)} = 0.05$	0.052	0.037	0.036	94	0.050	0.038	0.036	93
	$\theta_0^{(3)} = 0.08$	0.077	0.061	0.060	93	0.076	0.066	0.062	92